CS 7260 – INTERNET ARCHITECTURE AND PROTOCOLS
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On the last lecture, we had found that the expectation $E\left[Q_j(t)\right] = 1 - \gamma$. Then, we started working on another way to get a different value for $E\left[Q_j(t)\right]$, which is less straightforward, but hopefully, is different than $1 - \gamma$ and allow us to obtain the value of γ. If $N \to \infty$, then $D(t) \to \infty$ and $\frac{1}{N} \to 0$, hence, we also found that

$$Pr\left[A_j(t+1) = k\right] = e^{-\gamma} \cdot \frac{\gamma^k}{k!}$$

because it behaves like a Poisson distribution where $\lambda = \gamma$.

Now we will continue searching for the other definition of $E\left[Q_j(t)\right]$.

The renewal formula we are using is:

$$Q_j(t+1) = \max\left\{0, Q_j(t) + A_j(t+1) - 1\right\}$$

And we will try to figure out $E\left[Q_j(t)\right]$ from this equation.

We can define the sequence of $Q_j(t)$ as a stochastic process.

Stochastic Process: Sequence of random variable indexed by t (in this case).
Eventually, these random variables (in this case Poisson distributed) will be closer and closer to a distribution as t grows and thus closer to each other. So we want to figure out when $t \to \infty$, what $E\left[Q_j(t)\right]$ will be, and we will assume that all the variables have equal distribution, even though they are different random variables.

We will change notation now to:

$$Q_j(t+1) = \max\left\{0, Q_j(t) + A_j(t+1) - 1\right\} \to U = \max\{0, W + V - 1\}$$

Also, we will define the following probabilities:

$$u_i \triangleq Pr[U = i]$$
$$v_j \triangleq Pr[V = j]$$
$$w_k \triangleq Pr[W = k]$$

And thus, according to our renewal equation:

$$u_0 = v_1 w_0 + v_0 w_1 + v_0 w_0$$
$$u_1 = v_2 w_0 + v_1 w_1 + v_0 w_2$$
$$u_2 = v_3 w_0 + v_2 w_1 + v_1 w_2 + v_0 w_3$$
$$\ldots$$

And when $t \to \infty$ then $u_0 = v_0$; $u_1 = v_1$; $u_2 = v_2$; ... which turns the previous equations to

$$u_0 = u_1 w_0 + u_0 w_1 + u_0 w_0$$
$$u_1 = u_2 w_0 + u_1 w_1 + u_0 w_2$$
$$u_2 = u_3 w_0 + u_2 w_1 + u_1 w_2 + u_0 w_3$$
$$\ldots$$

We also have to consider that this transformation gives us:

$$1 = u_0 + u_1 + u_2 + u_3 + \cdots$$

and we have an infinite number linear equations with the "equally" infinite number of variables. We can solve this equations using a "generating function". To use the generating function (z transform), we will keep *u* and *v* different for now and we will replace them later.

Suppose:

$$U = V + W$$

Then we would have:

$$u_0 = v_0 w_0$$
$$u_1 = v_1 w_0 + v_0 w_1$$
$$u_2 = v_2 w_0 + v_1 w_1 + v_0 w_2$$
$$u_3 = v_3 w_0 + v_2 w_1 + v_1 w_2 + v_0 w_3$$
$$\ldots$$

which corresponds to a convolution and in the Z domain, it corresponds to:

$$U(z) = V(z) W(z)$$

When changing from time domain to Z domain (Generating Function) for some random discrete variable A(t) that takes value on non negative integers, we get:

$$A(z) \triangleq \sum_{i=0}^{\infty} a_i z^i$$

where,

$$a_i \triangleq Pr[A = i]$$

Now, for the "supposed" U(z), to demonstrate what it's Generating function looks like:

$$U(z) = \sum_{i=0}^{\infty} u_i z^i$$

$$U(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} v_i \, w_{i-j} z^j z^{i-j}$$

$$U(z) = \sum_{i=0}^{\infty} v_i z^i \sum_{j=0}^{\infty} w_j z^j = V(z) W(z)$$

So the "supposed" U(z), expanded looks like:

$$U(z) = V(z)W(z) = v_0w_0 + z(v_1w_0 + v_0w_1) + z^2(v_2w_0 + v_1w_1 + v_0w_2) + \cdots$$

To resolve the original equation $U = \max\{0, W + V - 1\}$, we will see the similarities with the "supposed" function $U = V + W$. We can see that all the terms on the original equation appear on the supposed equation but slightly displaced, such that for each u_i on the original equation, you have the terms of the u_{i+1} of the supposed equation, except on u_0 and u_1.

Now we will do black magic, and relate these 2 equations, to get the Generating Function for the original $U = \max\{0, W + V - 1\}$, using these similarities:

Applying Z transform to the "original" function we get:

$$\begin{aligned} U(z) &= v_0w_0 + v_1w_0 + v_0w_1 + z^1(v_2w_0 + v_1w_1 + v_0w_2) \\ &\quad + z^2(v_3w_0 + v_2w_1 + v_1w_2 + v_0w_3) + \cdots \end{aligned}$$

Now we will change our "original" U(z) for it to match the "supposed" U(z) and get its generating function. We will multiply all terms by z and add and sustract v_0w_0:

$$\begin{aligned} U(z) &= v_0w_0 + v_1w_0 + v_0w_1 + z^1(v_2w_0 + v_1w_1 + v_0w_2) \\ &\quad + z^2(v_3w_0 + v_2w_1 + v_1w_2 + v_0w_3) + \cdots \\ zU(z) &= z(v_0w_0) + z(v_1w_0 + v_0w_1) + z^2(v_2w_0 + v_1w_1 + v_0w_2) \\ &\quad + z^3(v_3w_0 + v_2w_1 + v_1w_2 + v_0w_3) + \cdots \\ zU(z) &= v_0w_0 - v_0w_0 + z(v_0w_0) + z(v_1w_0 + v_0w_1) + z^2(v_2w_0 + v_1w_1 + v_0w_2) \\ &\quad + z^3(v_3w_0 + v_2w_1 + v_1w_2 + v_0w_3) + \cdots \\ zU(z) &= z(v_0w_0) - v_0w_0 + v_0w_0 + z(v_1w_0 + v_0w_1) + z^2(v_2w_0 + v_1w_1 + v_0w_2) \\ &\quad + z^3(v_3w_0 + v_2w_1 + v_1w_2 + v_0w_3) + \cdots \\ zU(z) &= z(v_0w_0) - v_0w_0 + V(z)W(z) \\ zU(z) &= V(z)W(z) + (z-1)(v_0w_0) \end{aligned}$$

Now, this equation is equivalent to the infinite number of equations of the "original" U(z) described before.

Analyzing v_0w_0:
v_0w_0 = when this event happens, you are not getting throughput, so $v_0w_0 = 1 - \gamma$.
Also, we can now replace V by U and rearranging:

$$\begin{aligned} zU(z) &= V(z)W(z) + (z-1)(v_0w_0) \\ zU(z) &= U(z)W(z) + (z-1)(1-\gamma) \\ (z - W(z))U(z) &= (z-1)(1-\gamma) \end{aligned}$$

$$U(z) = \frac{(1-z)(1-\gamma)}{(W(z)-z)}$$

We know that W is Poisson distributed random variable so:

$$w_i \triangleq Pr[W = i] = e^{-\gamma} \cdot \frac{\gamma^i}{i!}$$

The Generating Function for W is obtained like this, note that we multiply by $e^{-\gamma z}$ and $e^{\gamma z}$:

$$W(z) = \sum_{i=0}^{\infty} w_i z^i$$

$$W(z) = \sum_{i=0}^{\infty} e^{-\gamma} \cdot \frac{\gamma^i}{i!} z^i = \sum_{i=0}^{\infty} e^{-\gamma} \cdot \frac{(\gamma z)^i}{i!} \cdot e^{-\gamma z} e^{\gamma z} = \sum_{i=0}^{\infty} (\frac{(\gamma z)^i}{i!} \cdot e^{-\gamma z}) \cdot e^{-\gamma} \cdot e^{\gamma z}$$

Since $\sum_{i=0}^{\infty} (\frac{(\gamma z)^i}{i!} \cdot e^{-\gamma z})$ should summate to 1, then:

$$W(z) = e^{-\gamma} \cdot e^{\gamma z} = e^{(z-1)\gamma}$$

And,

$$U(z) = \frac{(1-z)(1-\gamma)}{(e^{(z-1)\gamma} - z)}$$

Now, we want to get E[U] for this equation. In the Z domain, the E[U] is obtained by deriving over z and evaluating on z = 1 or on the limit where $z \rightarrow 1$ as we can see in the following equations:

$$E[U] = u_1 + u_2 \cdot 2 + u_3 \cdot 3 + \cdots = \sum_{i=0}^{\infty} i \cdot u_i$$

and,

$$U(z) = \sum_{i=0}^{\infty} u_i z^i$$

$$\frac{d}{dz} U(z) = \sum_{i=0}^{\infty} u_i \cdot i \cdot z^{i-1}$$

now if we evaluate at 1:

$$\frac{d}{dz} U(z)\bigg|_{z=1} = \sum_{i=0}^{\infty} u_i \cdot i \cdot z^{i-1}\bigg|_{z=1} = \sum_{i=0}^{\infty} i \cdot u_i$$

Which leaves clear how to get E[U] from a generating function.

Now z = 1 will give us $\frac{0}{0}$ in our U(z) so we will use $z \to 1$ and we will use L'Hospital.

$$U(z)|_{z\to 1} = \left.\frac{(1-z)(1-\gamma)}{(e^{(z-1)\gamma} - z)}\right|_{z\to 1} = E[U]$$

When we resolve this derivation we will find that:

$$E[U] = \frac{\gamma^2}{2(1-\gamma)} = E\left[Q_j(t)\right]$$

Now, we will use the previous equation of $E\left[Q_j(t)\right] = 1 - \gamma$ to get γ.

$$\frac{\gamma^2}{2(1-\gamma)} = 1 - \gamma$$

$$\gamma = 2 \pm \sqrt{2}$$

The $\gamma = 2 + \sqrt{2}$ is not valid, since that would be a saturation over 1 and we will use the $\gamma = 2 - \sqrt{2} = 56.8\%$ which is what we were looking for.

The $1 - e^{-1}$ Value

There is another important quantity $1 - e^{-1} = 63.2\%$.

The $1 - e^{-1}$ value comes from the fact that for an arbitrary output port j, the probability of having a packet sent to it if all output ports have equal probability of receiving a packet is:

$$\frac{1}{N}$$

and the probability of not having any packet sent to it will be

$$1 - \frac{1}{N}$$

Now, for all N input ports, the probability that no packets are delivered to it is:

$$(1 - \frac{1}{N})^N$$

And when $N \to \infty$, this value will be: e^{-1}, and this probability will basically be how much through put is lost, because if this event happens, the corresponding output port will not be used. So, the thorough put value will be:

$$1 - e^{-1}$$

This important quantity appears when we have fresh start on every input port after every departure. Which means that each input port will have an equal probability of getting cells to every output port on any cycle.

The reason we get 56.8% instead of 63.2% is because this situation is not real. If a packet was not delivered in one cycle, it will stay as the HOL on that input queue in the next cycle, and so it will have a stronger tendency to collide with other HOL packets going to the same output port. This means the probability of the packet going to a specific output port on the next cycle, will not be equal for all output ports, and the assumption we were making doesn't happen because a given port j will not have $\frac{1}{N}$ probability of receiving a packet.