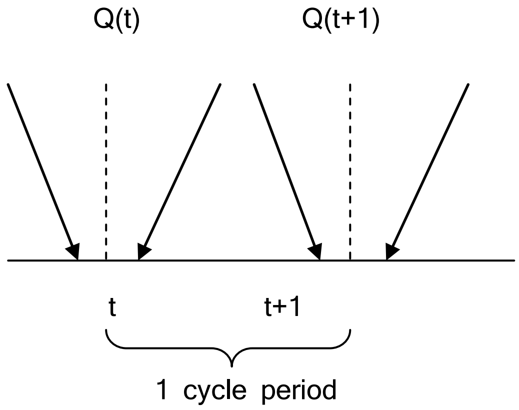
Oct 24, 2012 *Ankita Mathur*



$$Qj(t+1) = \max\{0,\ Qj(t) + Aj(t+1) - 1\}$$ **Renewal Equation**

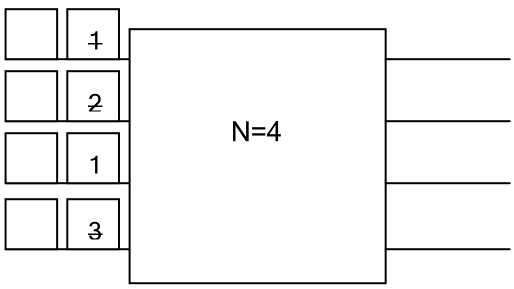
- Qj(t) - # of HOL cells at 'N' input ports destined for output port 'j'

Important characteristic of Qj(t) is that switching happens just before that and an arrival occurs after that which is called Aj(t+1).

Thus Qj(t) is the queue length at time 't' and Aj(t+1) represent the cells that arrive after that at time t+e. If the sum of these 2 terms is >0, then there will be a packet switched before Qj(t+1) (i.e. a departure).

Therefore,

queue_length at t+1 = (queue_length at t) + (any arrivals) - (departure)



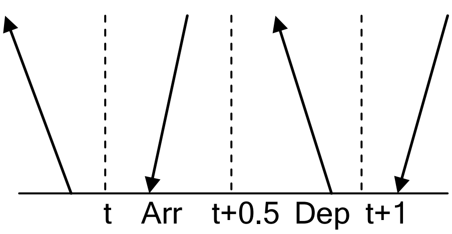
Q1(t+1) = 1

Q2(t+1) = 0

Q3(t+1) = 0

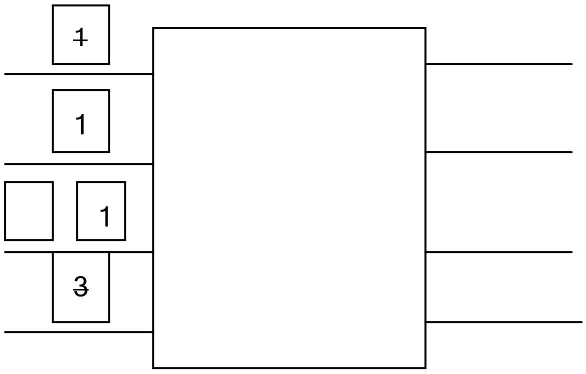
state at time 't+1'

t+0.5



At cycle t+0.5, the state would be as shown in the figure with cells switched for output ports 1,2 and 3.

At cycle t+1.5



Q1(t+2) = 2

Q2(t+2) =

Q4(t+2) =

Q3(t+2) = 0

state at time 't+2'

Ideal situation:

For all j, Qj(t) = 0. i.e. all the input packets have been switched. If all are going to distinct ports, there will be N arrivals and N departures.

However, it is not possible to obtain ideal throughput because of collisions.

Let the saturation throughput be γ

Let the expectation of Qj(t), E[Qj(t)] = ?

There are N HOL cells to start with and D(t) departures at time 't'.

$Q(t) = N - D(t)$

Therefore, $Q(t)/N = 1 - D(t)/N$

$$\downarrow$$

$$Q(t)/N = 1 - \gamma$$

Now $Q(t) = \sum_{j=1}^{N} Q_j(t)$

Therefore, $\left(\sum_{j=1}^{N} Q_j(t) \right)/N = 1 - \gamma$

Calculating the Expectation on each side,

$E[(\sum_{j=1}^{N} Q_j(t))/N] = E[1 - \gamma]$

Expectation of $1-\gamma$ is $1-\gamma$ itself.

Every inout port are identical and thus have equal distribution. Therefore,

$$E[(\sum_{j=1}^{N} Q_j(t))/N] = 1/N * E[(\sum_{j=1}^{N} Q_j(t)]$$

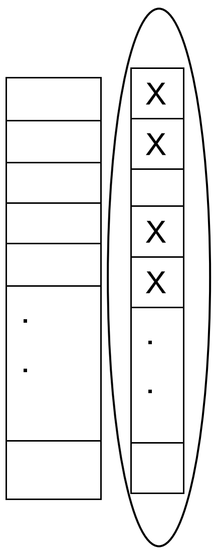
$$= 1/N * N * E[Q_j(t)]$$

$$= E[Q_j(t)]$$

Therefore, $\boxed{1 - \gamma = E[Q_j(t)]}$

Alternate formula for $E[Q_j(t)]$ for calculating the value of γ

$D(t)$ = # departures at time $t - \varepsilon$



D(t) number of these HOL cells are gone. Among these D(t) cells, some may go to output port 1, some to output port 2 and so on. This distribution of cells at each output port follows the Binomial Distribution. Since there were D(t) departures, there will be D(t) arrivals. Among these D(t) arrivals say,

$A_j(t+1)$ cells will go to output port 'j'.

Therefore, $D(t) = \sum_{j=1}^{N} A_j(t+1)$ arrivals.

Consider a fixed 'j' and when a steady state is reached, $\lim_{t\to\infty} \lim_{N\to\infty} D(t)/N = \gamma$ which is the saturation throughput.

Therefore, the $Pr[A_j(t+1) = k] = \binom{D(t)}{k}(\frac{1}{N})^k(1-\frac{1}{N})^{D(t)-k}$

where $k = 0,1,2,3..D(t)$

Recall, for a Binomial distribution, $\lim_{\substack{n \to \infty \\ p \to \infty \\ np \to \lambda}} B(n,\ p) = \text{Poisson}(\lambda)$

Therefore, Considering the Binomial equation Binomial(D(t), 1/N) -> Possion(γ)

and rewriting $\Pr[\ A_j(t+1) = k] = e^{-\gamma} (\gamma^{k} / k!)$

This means that the number of arrivals at a particular input port 'j', $A_j(t+1)$ is a random variable and follows the Poisson distribution.

Thus, the Renewal equation is a well-define queueing process. The 'Arrivals' follow Poisson Distibution with parameter γ, there is one departure. And with this, the average queue size can be determined.

Solving the Renewal equation, the value of $E[Q_j(t)]$ can be determined and it is

$$E[Q_j(t)] = \gamma^2 / 2(1-\gamma)$$

Therefore equating this with $(1-\gamma)$, we obtain

$$\boldsymbol{\gamma^2 / 2(1-\gamma) = (1-\gamma)}$$

$$\boldsymbol{\gamma = 2 - \sqrt{2}}$$