Topics:

- Reinforcement Learning Part 1
  - **Markov Decision Processes**
  - **Value Iteration**

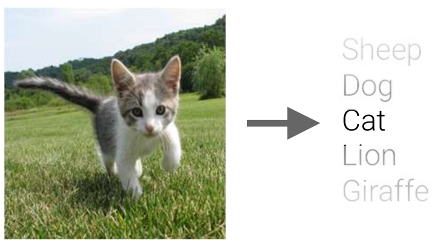
CS 4803-DL / 7643-A

ZSOLT KIRA

Reinforcement Learning Introduction

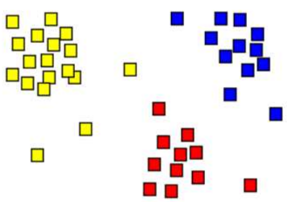
Supervised Learning

- Train Input: $\{X, Y\}$
- Learning output: $f : X \rightarrow Y,\ P(y|x)$
- e.g. classification



Unsupervised Learning

- Input: $\{X\}$
- Learning output: $P(x)$
- Example: Clustering, density estimation, etc.



Reinforcement Learning

- Evaluative feedback in the form of **reward**
- No supervision on the right action

Types of Machine Learning

RL: Sequential decision making in an environment with evaluative feedback.

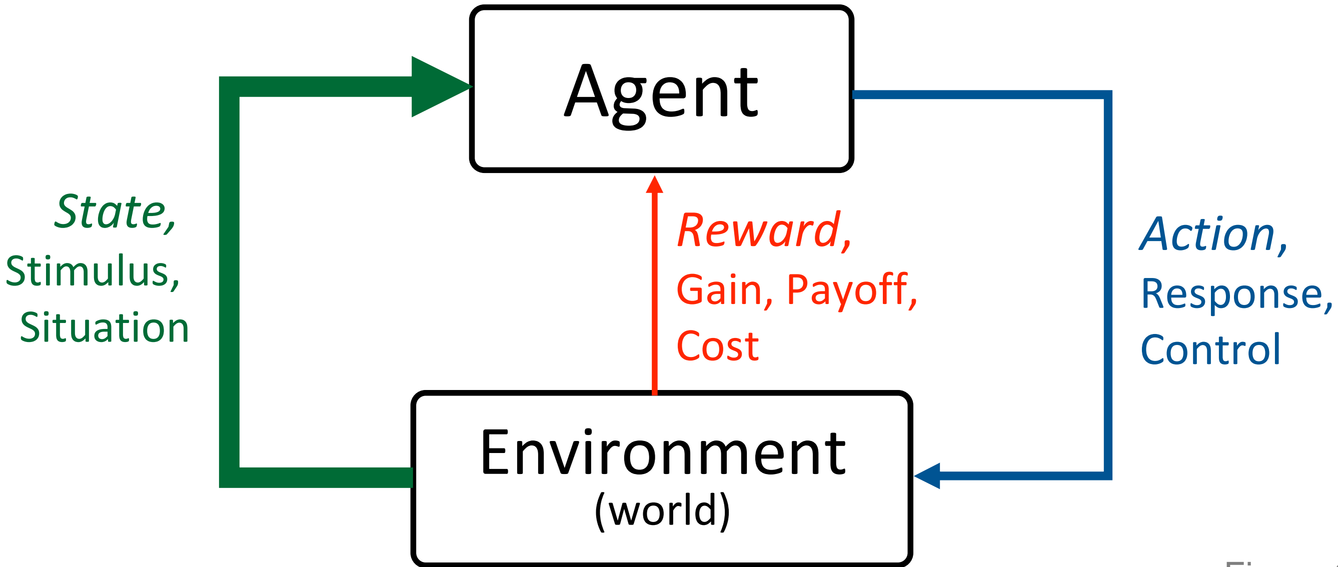


Figure Credit: Rich Sutton

- **Environment** may be unknown, non-linear, stochastic and complex.
- **Agent** learns a **policy** to map states of the environments to actions.
  - Seeking to maximize cumulative reward in the long run.

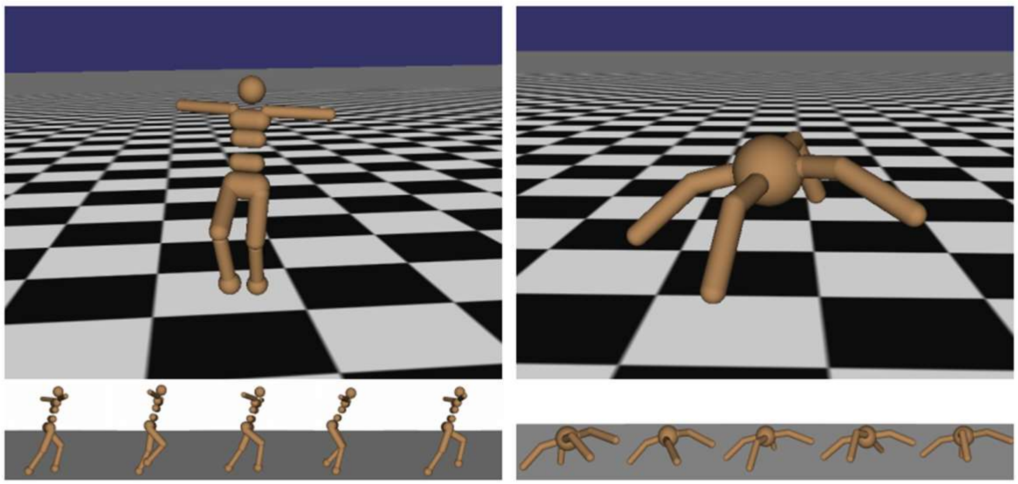
What is Reinforcement Learning?

Signature Challenges in Reinforcement Learning

- Evaluative feedback: Need trial and error to find the right action
- Delayed feedback: Actions may not lead to immediate reward
- Non-stationarity: Data distribution of visited states changes when the policy changes
- Fleeting nature of time and online data

Slide adapted from: Richard Sutton

Robot Locomotion



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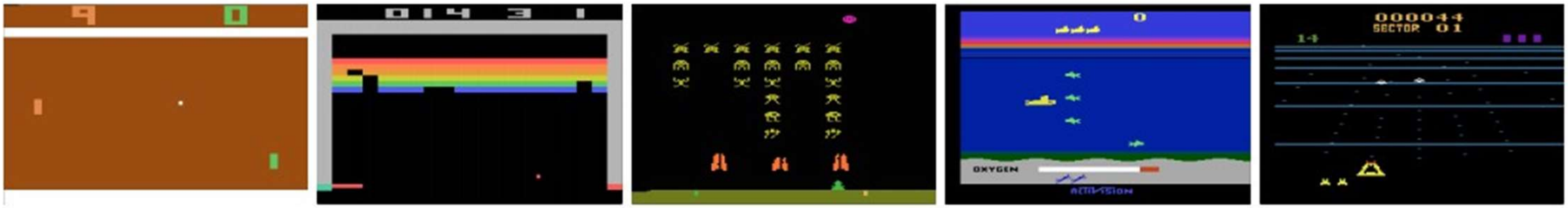
- **Objective**: Make the robot move forward
- **State**: Angle and position of the joints
- **Action**: Torques applied on joints
- **Reward**: +1 at each time step upright and moving forward

Slide Credit: Fei-Fei Li, Justin Johnson, Serena Yeung, CS 231n

Atari Games

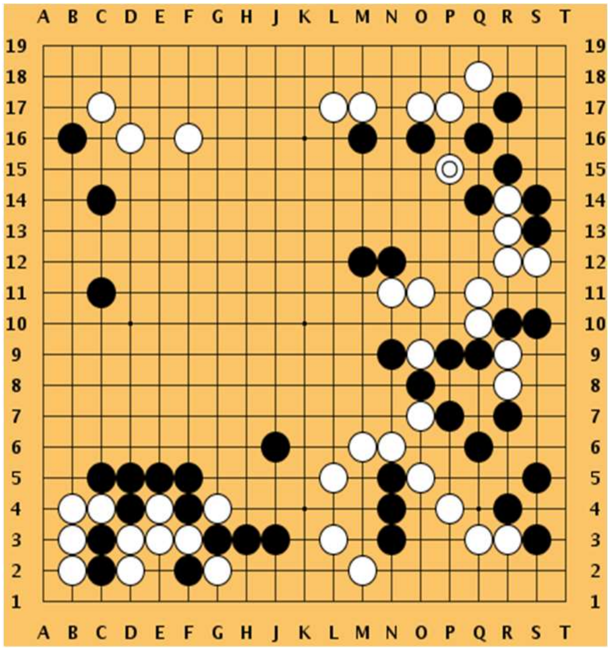


- **Objective**: Complete the game with the highest score
- **State**: Raw pixel inputs of the game state
- **Action**: Game controls e.g. Left, Right, Up, Down
- **Reward**: Score increase/decrease at each time step

Slide Credit: Fei-Fei Li, Justin Johnson, Serena Yeung, CS 231n

Examples of RL tasks

Go



- **Objective**: Defeat opponent
- **State**: Board pieces
- **Action**: Where to put next piece down
- **Reward**: +1 if win at the end of game, 0 otherwise

Slide Credit: Fei-Fei Li, Justin Johnson, Serena Yeung, CS 231n

Markov Decision Processes

- **MDPs**: Theoretical framework underlying RL

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- An MDP is defined as a tuple $(\mathcal{S}, \mathcal{A}, \mathcal{R}, \mathbb{T}, \gamma)$
  - $\mathcal{S}$: Set of possible states
  - $\mathcal{A}$: Set of possible actions
  - $\mathcal{R}(s, a, s')$: Distribution of reward
  - $\mathbb{T}(s, a, s')$: Transition probability distribution, also written as p(s'|s,a)
  - γ : Discount factor

Markov Decision Processes (MDPs)

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- **Interaction trajectory**: $\ldots s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1}, r_{t+2}, s_{t+2}, \ldots$
- **Markov property**: Current state completely characterizes state of the environment
- **Assumption**: Most recent observation is a sufficient statistic of history

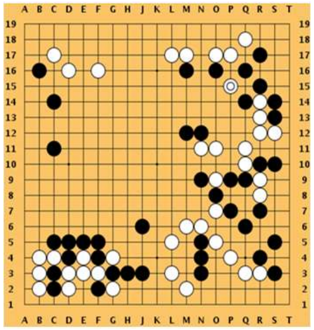
$$p\left(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, \ldots S_0 = s_0\right) = p\left(S_{t+1} = s' | S_t = s_t, A_t = a_t\right)$$

Markov Decision Processes (MDPs)

MDP Variations

Fully observed MDP

- Agent receives the true state s_t at time t
- Example: Chess, Go



Partially observed MDP

- Agent perceives its own partial observation o_t of the state s_t at time t, using past states e.g. with an RNN
- Example: Poker, First-person games (e.g. Doom)

Source: https://github.com/mwydmuch/ViZDoom

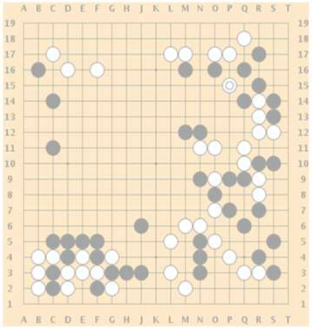
Fully observed MDP

- Agent receives the true state s_t at time t
- Example: Chess, Go

Partially observed MDP

- Agent perceives its own partial observation o_t of the state s_t at time t, using past

We will assume **fully observed MDPs** for this lecture



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MDP Variations

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- In **Reinforcement Learning**, we assume an underlying **MDP** with unknown:
  - Transition probability distribution $\mathbb{T}$
  - Reward distribution $\mathcal{R}$

MDP
$(\mathcal{S}, \mathcal{A}, \mathcal{R}, \mathbb{T}, \gamma)$

MDPs in the context of RL

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- Evaluative feedback comes into play, trial and error necessary

- For this lecture, assume that we know the true reward and transition distribution and look at algorithms for **solving MDPs** i.e. finding the best policy
  - Rewards known everywhere, no evaluative feedback
  - Know how the world works i.e. all transitions

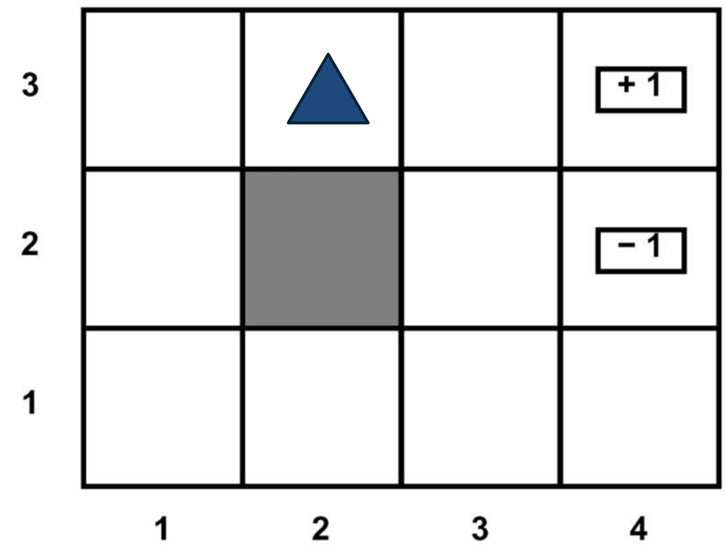


Figure credits: Pieter Abbeel

A Grid World MDP

- Agent lives in a 2D grid environment

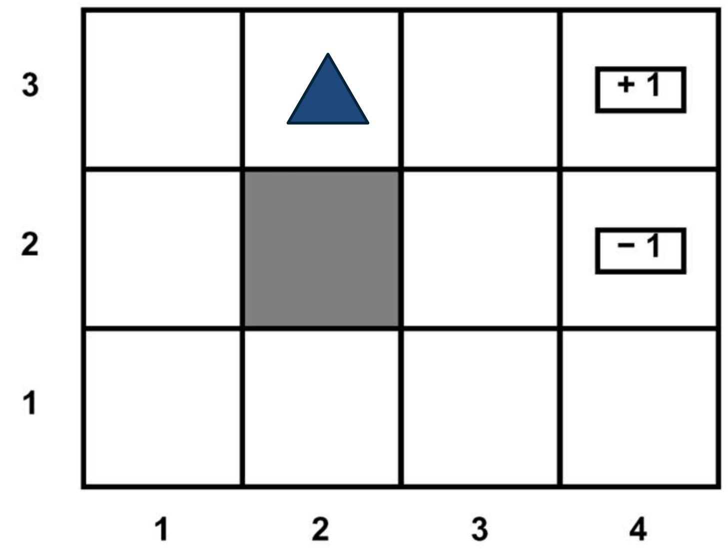
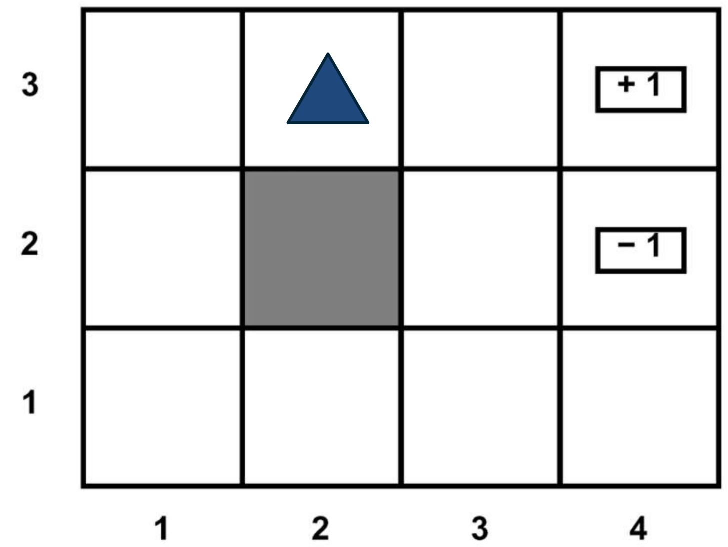


Figure credits: Pieter Abbeel

A Grid World MDP

- Agent lives in a 2D grid environment
- State: Agent's 2D coordinates
- Actions: N, E, S, W
- Rewards: +1/-1 at absorbing states



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- Agent lives in a 2D grid environment

- State: Agent's 2D coordinates
- Actions: N, E, S, W
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- Walls block agent's path
- Actions to not always go as planned
  - 20% chance that agent drifts one cell left or right of direction of motion (except when blocked by wall).

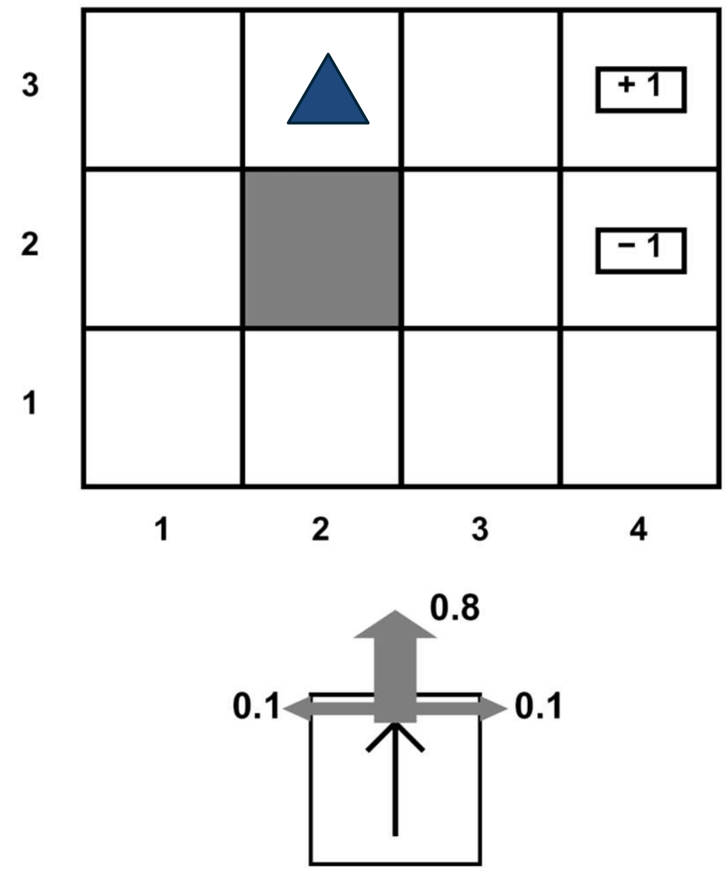

Figure credits: Pieter Abbeel

- Solving MDPs by finding the **best/optimal policy**

Solving MDPs: Optimal policy

- Solving MDPs by finding the **best/optimal policy**
- Formally, a **policy** is a mapping from states to actions

e.g.

State	Action
A →	2
B →	1

Solving MDPs: Optimal policy

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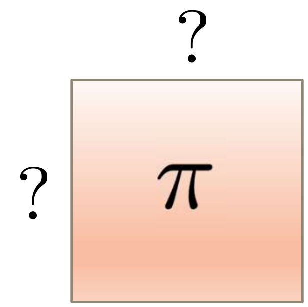
- Solving MDPs by finding the **best/optimal policy**

- Formally, a **policy** is a mapping from states to actions
  - Deterministic $\pi(s) = a$

$$n = |\mathcal{S}|$$

$$m = |\mathcal{A}|$$



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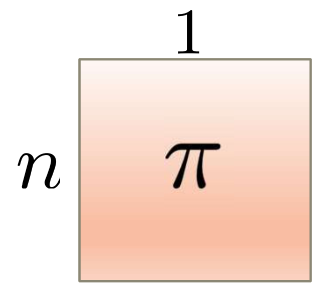

Solving MDPs: Optimal policy

- Solving MDPs by finding the **best/optimal policy**

- Formally, a **policy** is a mapping from states to actions
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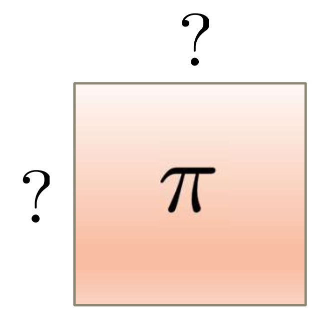

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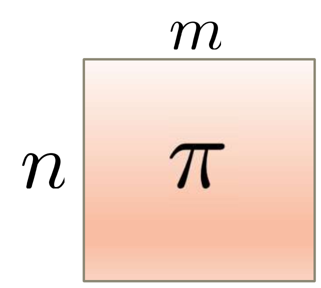

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- What is a good policy?
  - Maximize **current reward**? Sum of all **future rewards**?
  - **Discounted sum of future rewards**!
    - Discount factor: γ

Solving MDPs: Optimal policy

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- Formally, the **optimal policy** is defined as:

$$\pi^* = \arg\max_{\pi} \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r_t | \pi\right]$$

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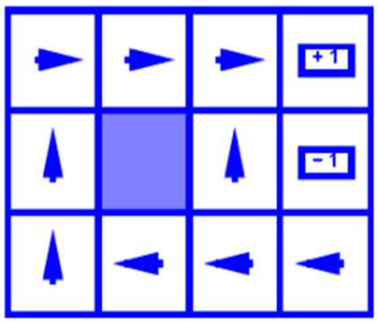
$$\boldsymbol{s}_0 \sim p\left(\boldsymbol{s}_0\right), a_t \sim \pi\left(\cdot|\boldsymbol{s}_t\right), \boldsymbol{s}_{t+1} \sim p\left(\cdot|\boldsymbol{s}_t, a_t\right)$$

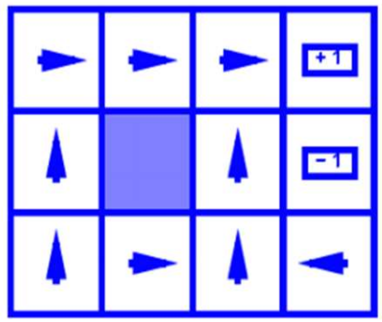
Expectation over initial state, actions from policy,
next states from transition distribution

Solving MDPs: Optimal policy

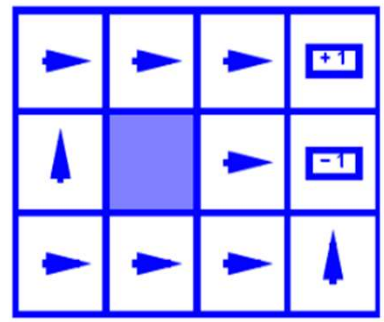
- Some optimal policies for three different grid world MDPs (gamma=0.99)
    - Varying reward for non-absorbing states (states other than +1/-1)



R(s) = -0.03



R(s) = -0.4



R(s) = -2.0

Image Credit: Byron Boots, CS 7641

Optimal policy examples

- A **value function** is a prediction of discounted sum of future reward

Value Function

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- **State** value function / **V**-function / $V : \mathcal{S} \rightarrow \mathbb{R}$

Value Function

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- A **value function** is a prediction of discounted sum of future reward

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  - How good is this state?
  - Am I likely to win/lose the game from this state?

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- **State-Action** value function / **Q**-function / $Q : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$
  - How good is this state-action pair?
  - In this state, what is the impact of this action on my future?

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- For a policy that produces a trajectory sample $(s_0, a_0, s_1, a_1, s_2 \cdots)$

Value Function

- For a policy that produces a trajectory sample $(s_0, a_0, s_1, a_1, s_2 \cdots)$

- The **V-function** of the policy at state s, is the expected cumulative reward from state s:

$$V^{\pi}(s) = \mathbb{E}\left[\sum_{t \geq 0} \gamma^t r_t | s_0 = s, \pi\right]$$

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Georgia Tech

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↑ ?

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Value Function

- For a policy that produces a trajectory sample $(s_0, a_0, s_1, a_1, s_2 \cdots)$

- The **Q-function** of the policy at state **s** and action **a**, is the expected cumulative reward upon taking action **a** in state **s** (and following policy thereafter):

Action-Value Function

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$$Q^{\pi}(s,a) = \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r_t | s_0 = s, a_0 = a, \pi\right]$$

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Action-Value Function

- The V and Q functions corresponding to the optimal policy $\pi^\star$

$$V^*(s) = \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r_t | s_0 = s, \pi^*\right]$$

$$Q^*(s,a) = \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r_t | s_0 = s, a_0 = a, \pi^*\right]$$

Optimal V & Q functions

Recursive Bellman expansion (from definition of Q)

(Expected) return from t = 0

$$Q^*(s,a) = \mathbb{E}_{\substack{a_t \sim \pi^*(\cdot|s_t) \\ s_{t+1} \sim p(\cdot|s_t,a_t)}} \left[\sum_{t \geq 0} \gamma^t r(s_t, a_t) \mid s_0 = s, a_0 = a \right]$$

(Reward at t = 0) + gamma * (Return from expected state at t=1)

$$= \gamma^0 r(s,a) + \mathbb{E}_{s' \sim p(\cdot|s,a)} \left[\gamma \mathbb{E}_{\substack{a_t \sim \pi^*(\cdot|s_t) \\ s_{t+1} \sim p(\cdot|s_t,a_t)}} \left[\sum_{t \geq 1} \gamma^{t-1} r(s_t, a_t) \mid s_1 = s' \right] \right]$$

$$= r(s,a) + \gamma \mathbb{E}_{s' \sim p(s'|s,a)} [V^*(s')]$$

$$= \mathbb{E}_{s' \sim p(s'|s,a)} [r(s,a) + \gamma V^*(s')]$$

Bellman Optimality Equations

- Equations relating optimal quantities

$$V^*(s) = \max_a Q^*(s,a)$$

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

- Recursive Bellman optimality equation

$$\begin{aligned} Q^*(s,a) &= \mathbb{E}_{s' \sim p(s'|s,a)} \left[r(s,a) + \gamma V^*(s') \right] \\ &= \sum_{s'} p(s'|s,a) \left[r(s,a) + \gamma V^*(s') \right] \\ &= \sum_{s'} p(s'|s,a) \left[r(s,a) + \gamma \max_{a} Q^*(s',a') \right] \end{aligned}$$

NOTE: In the lecture video for these slides, there was a typo having V(s) instead of V(s')

- Equations relating optimal quantities

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$$V^*(s) = \max_a \sum_{s'} p(s'|s,a) \left[r(s,a) + \gamma V^*(s') \right]$$

Bellman Optimality Equations

Based on the **bellman optimality equation**

$$V^*(s) = \max_a \sum_{s'} p(s'|s,a)\left[r(s,a) + \gamma V^*(s')\right]$$

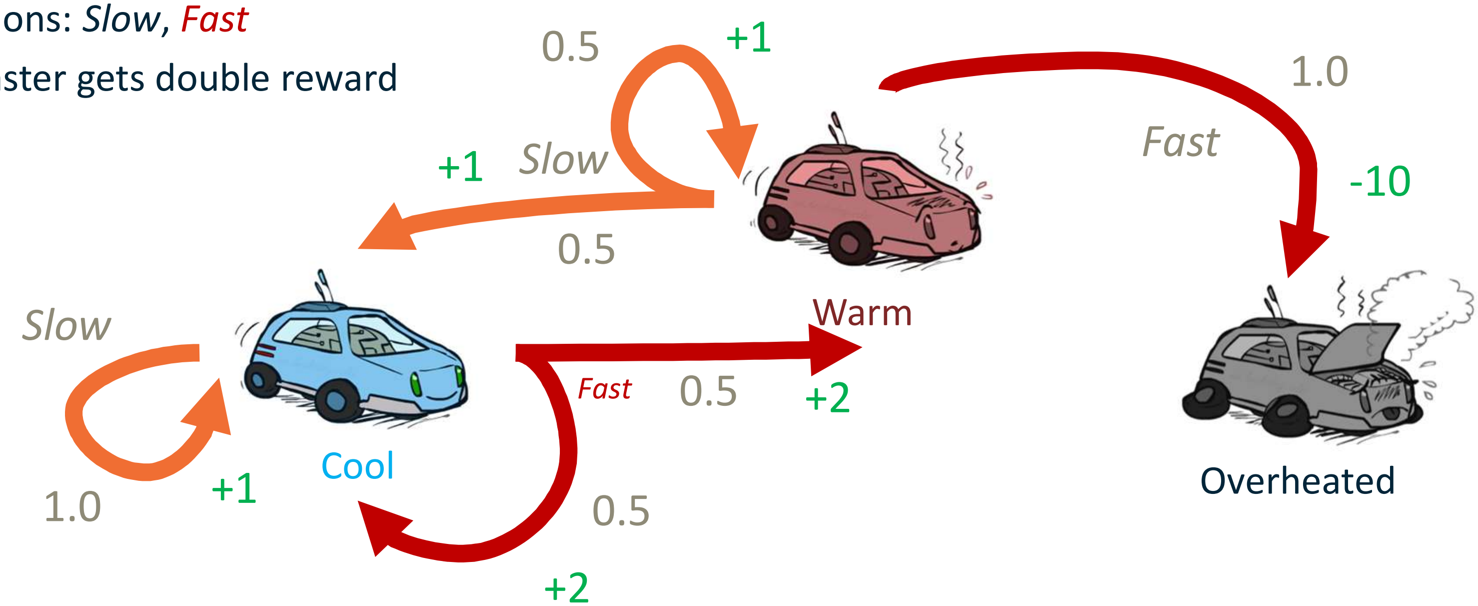
Algorithm

- Initialize values of all states
- While not converged:
  - For each state: $V^{i+1}(s) \leftarrow \max_a \sum_{s'} p(s'|s,a)\left[r(s,a) + \gamma V^i(s')\right]$
- Repeat until convergence (no change in values)

$$V^0 \rightarrow V^1 \rightarrow V^2 \rightarrow \quad \cdots \rightarrow V^i \rightarrow \ldots \quad \rightarrow V^*$$

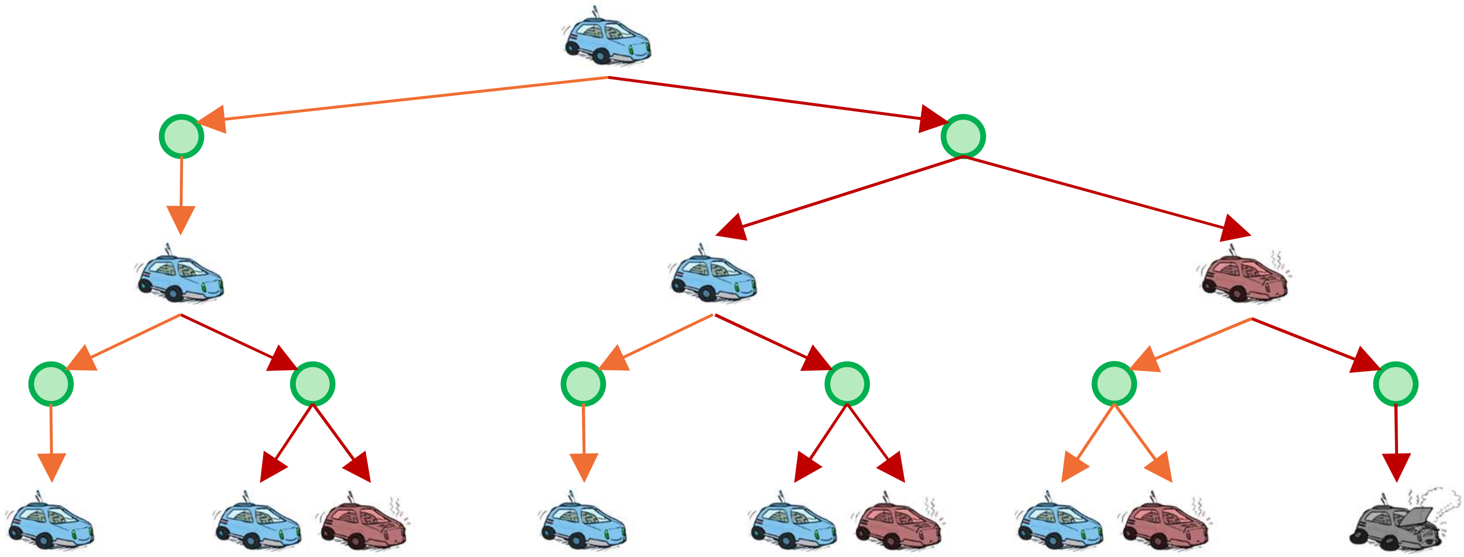
Time complexity per iteration $O(|\mathcal{S}|^2|\mathcal{A}|)$

- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward



Slide Credit: http://ai.berkeley.ed

Example: Racing



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Racing Search Tree

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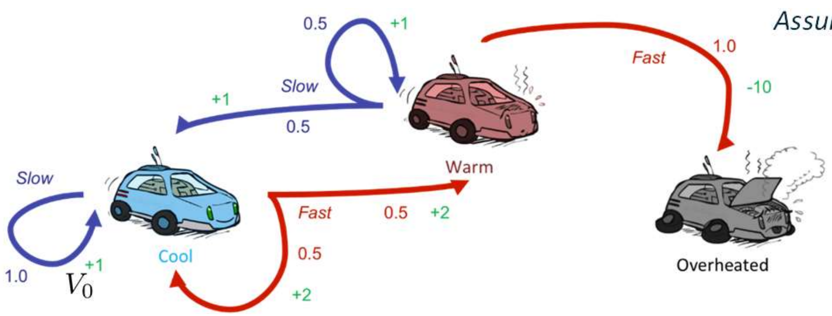

Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

V_2	3.5	2.5	0
V_1	2	1	0
	0	0	0

Slide Credit: http://ai.berkeley.edu

Racing Search Tree

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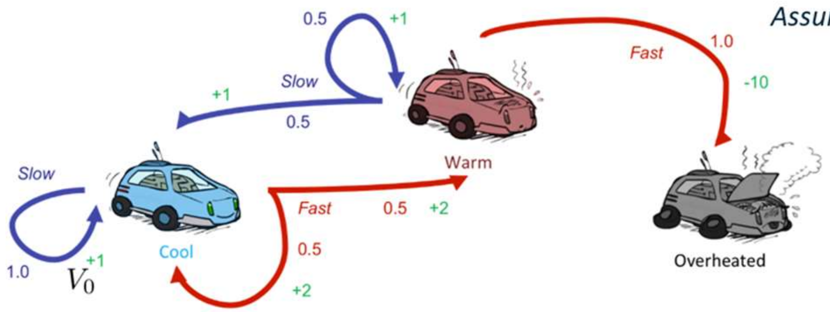


V_0

Assume no discount!

V_2			
V_1	2.		
	0	0	0

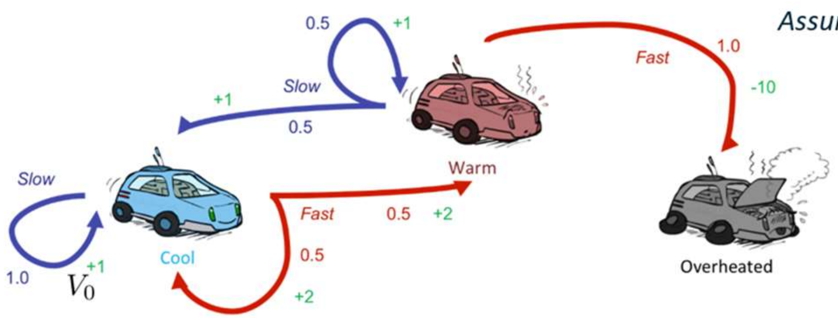
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$

$V_1(s_0) = \max(\,$ slow: $[1.0 \cdot (+1 + V(s_0))] = 1,$

Fast: $0.5[+2 + V(s_1)] + 0.5(+2 + V(s_0))] = 2\,)$

$= 2$

Slide Credit: http://ai.berkeley.edu



Assume no discount!

V_2	3.5	2.5	0
V_1	2	1	0
	0	0	0

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

$$V_2(s_0) = \max \Big[\underbrace{1 \cdot (+1 + V_1(s_0))}_{\text{slow}} = 3,$$
$$\underbrace{0.5\,(+2 + V_1(s_0)) + 0.5\,(+2 + V_1(s_1))}_{\text{fast}} = 3.5 \Big]$$

($V_1(s_0) = 2$, $V_1(s_1) = 1$)

$= 3.5$

Slide Credit: http://ai.berkeley.edu

Racing Search Tree

Georgia Tech

Value Iteration Update:

$$V^{i+1}(s) \leftarrow \max_a \sum_{s'} p(s'|s,a)\left[r(s,a) + \gamma V^i(s')\right]$$

Q-Iteration Update:

$$Q^{i+1}(s,a) \leftarrow \sum_{s'} p\left(s'|s,a\right)\left[r\left(s,a\right) + \gamma \max_{a'} Q^i(s',a')\right]$$

The algorithm is same as value iteration, but it loops over actions as well as states

Q-Iteration

For **Value Iteration**:

Theorem: will converge to unique optimal values
Basic idea: approximations get refined towards optimal values
Policy may converge long before values do

Time complexity per iteration $O(|\mathcal{S}|^2|\mathcal{A}|)$

Feasible for:

- 3x4 Grid world?
- Chess/Go?
- Atari Games with integer image pixel values [0, 255] of size 16x16 as state?

Summary: MDP Algorithms

Value Iteration

- Bellman update to state value estimates

Q-Value Iteration

- Bellman update to (state, action) value estimates