Topics:

- Linear Classification, Loss functions
- Gradient Descent

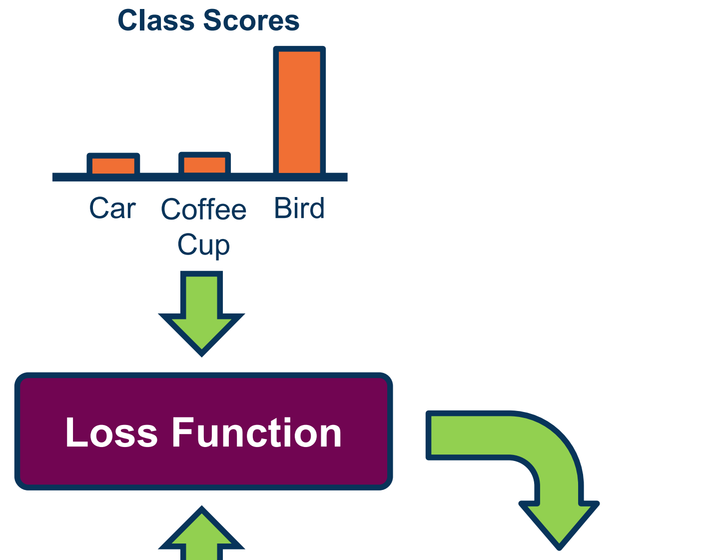
CS 4644-DL / 7643-A
ZSOLT KIRA

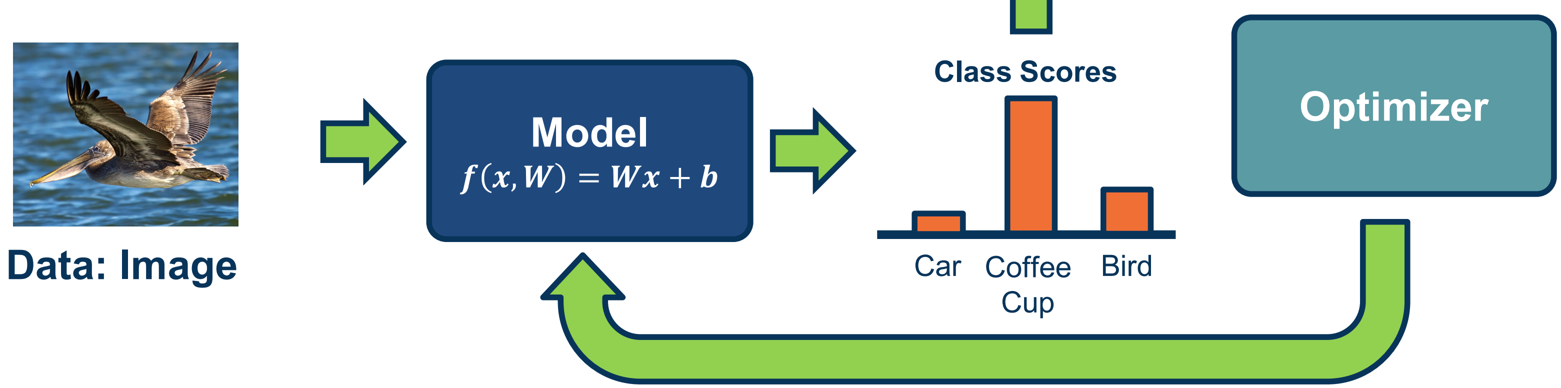
- **Assignment 1 out!**
  - Due Feb. 3rd
  - Start early, start early, start early!
  - HW1 Tutorial: Monday
  - Matrix Calculus Tutorial: next Thursday

- **Piazza**: Enroll now! (Code: DLSPR2022)
  - **NOTE:** There is an OMSCS section with a Ed. Make sure you are in the right one

- **Office hours** schedule: https://piazza.com/class/lcl94yjxkbb59e/post/59
  - Use canvas zoom schedule

- Input (and representation)
- Functional form of the model
    - Including parameters
- Performance measure to improve
    - Loss or objective function
- Algorithm for finding best parameters
    - Optimization algorithm





Components of a Parametric Model

$$f(x, w) = y$$

Classifier · **Input (vector)** · **Weights** · **Output (scalar or vector)**

- **Input:** Continuous number or vector
- **Output:** A continuous number
  - For classification typically a **score**
  - For regression what we want to regress to (house prices, crime rate, etc.)
- w **is a vector and weights** to optimize to fit target function

Model: Discriminative Parameterized Function

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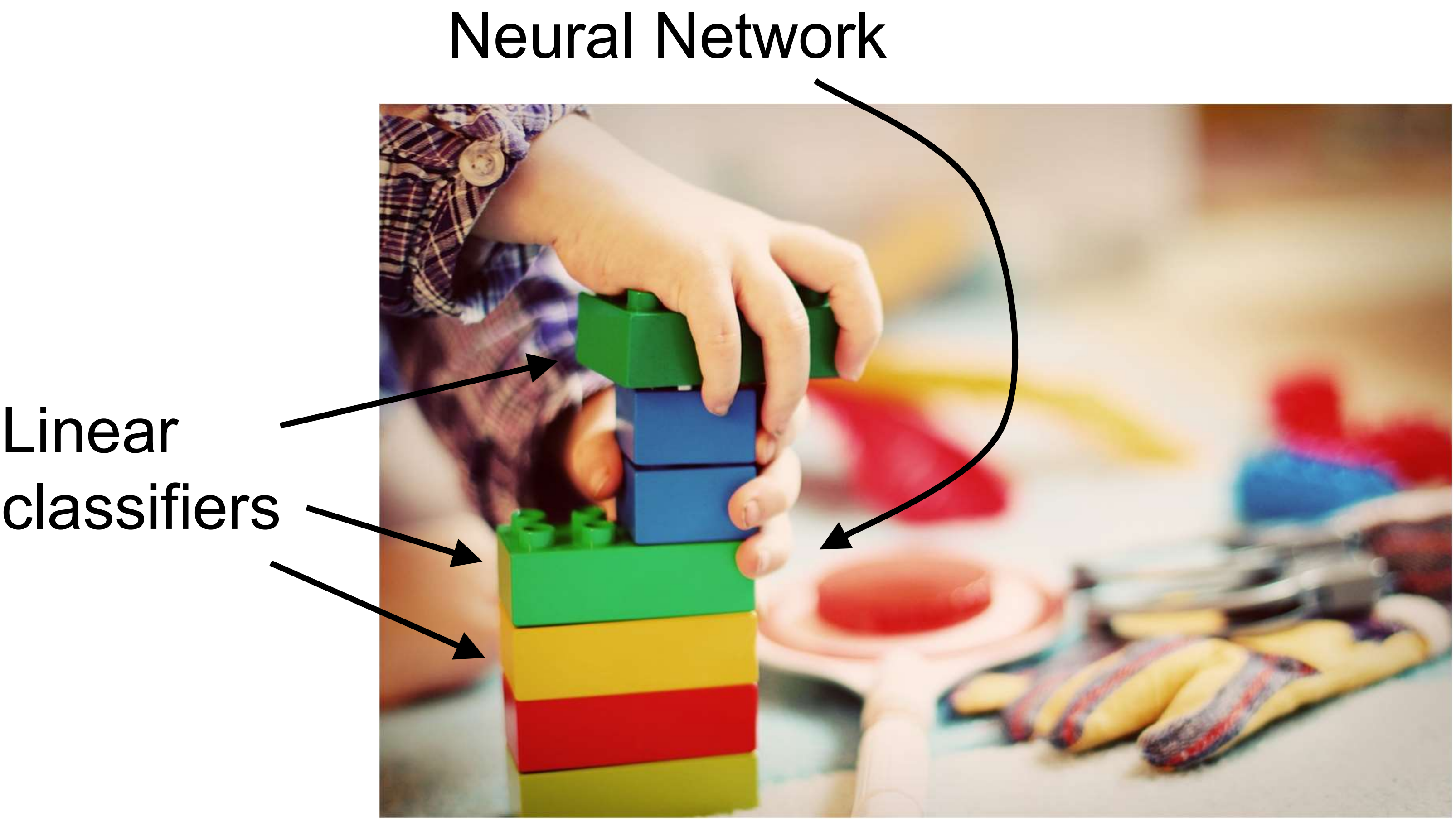
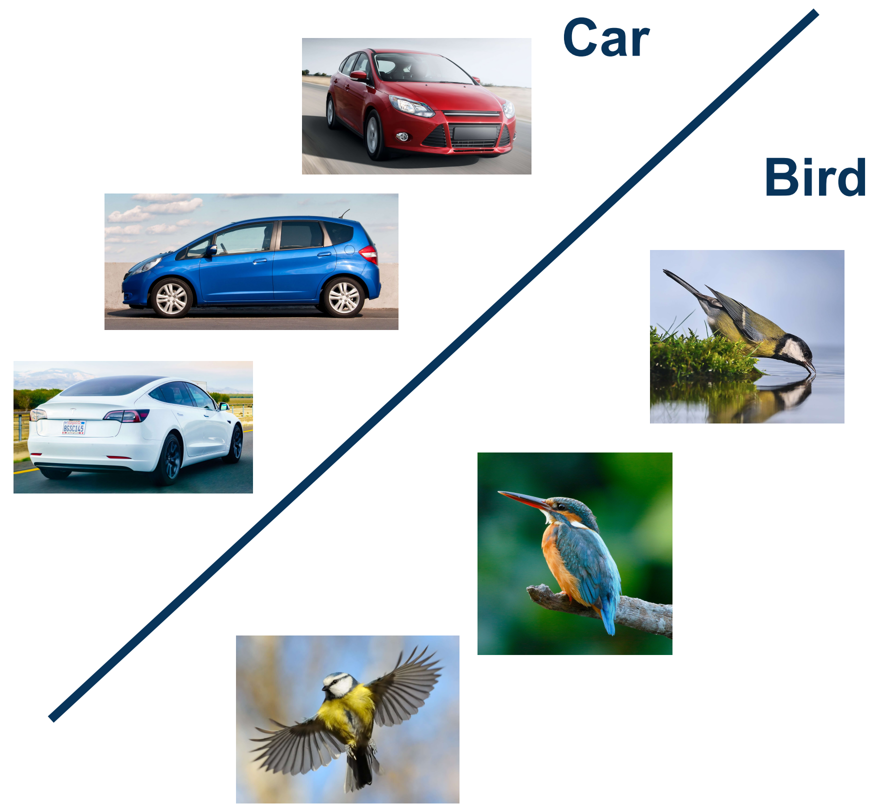


Slide Credit: Fei-Fei Li, Justin Johnson, Serena Yeung, CS 231n

Deep Learning as Legos

- **Idea:** Separate classes via high-dimensional linear separators (hyper-planes)
- One of the simplest parametric models, **but surprisingly effective**
  - Very commonly used!
- Let's look more closely at each element

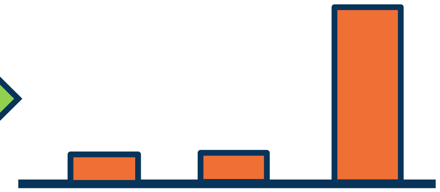


Data: Image

Model

$f(x, W) = Wx + b$

Class Scores



$$x = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$$

Flatten

$$x = \begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{21} \\ x_{22} \\ \vdots \\ x_{n1} \\ \vdots \\ x_{nn} \end{bmatrix}$$

To simplify notation we will refer to inputs as $x_1 \cdots x_m$ where $m = n \times n$

Input Dimensionality

Model

$$f(x, W) = Wx + b$$

Classifier for class 1 →
Classifier for class 2 →
Classifier for class 3 →

$$\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1m} \\ w_{21} & w_{22} & \cdots & w_{2m} \\ w_{31} & w_{32} & \cdots & w_{3m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\boldsymbol{W}$ $\boldsymbol{x}$ $\boldsymbol{b}$

(Note that in practice, implementations can use xW instead, assuming a different shape for W. That is just a different convention and is equivalent.)

Weights

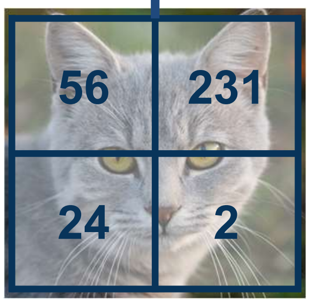
- We can move the bias term into the weight matrix, and a “1” at the end of the input
- Results in **one matrix-vector multiplication**!

Model
$f(x, W) = Wx + b$

$$\underbrace{\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1m} & b_1 \\ w_{21} & w_{22} & \cdots & w_{2m} & b_2 \\ w_{31} & w_{32} & \cdots & w_{3m} & b_3 \end{bmatrix}}_{\boldsymbol{W}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ 1 \end{bmatrix}}_{\boldsymbol{x}}$$

Example with an image with **4 pixels**, and **3 classes** (cat/dog/ship)

Stretch pixels into column



Input image

$$\underbrace{\begin{bmatrix} 0.2 & -0.5 & 0.1 & 2.0 \\ 1.5 & 1.3 & 2.1 & 0.0 \\ 0 & 0.25 & 0.2 & -0.3 \end{bmatrix}}_{W} \begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1.1 \\ 3.2 \\ -1.2 \end{bmatrix}}_{b} = \begin{bmatrix} -96.8 \\ 437.9 \\ 61.95 \end{bmatrix} \begin{matrix} \text{Cat score} \\ \text{Dog score} \\ \text{Ship score} \end{matrix}$$

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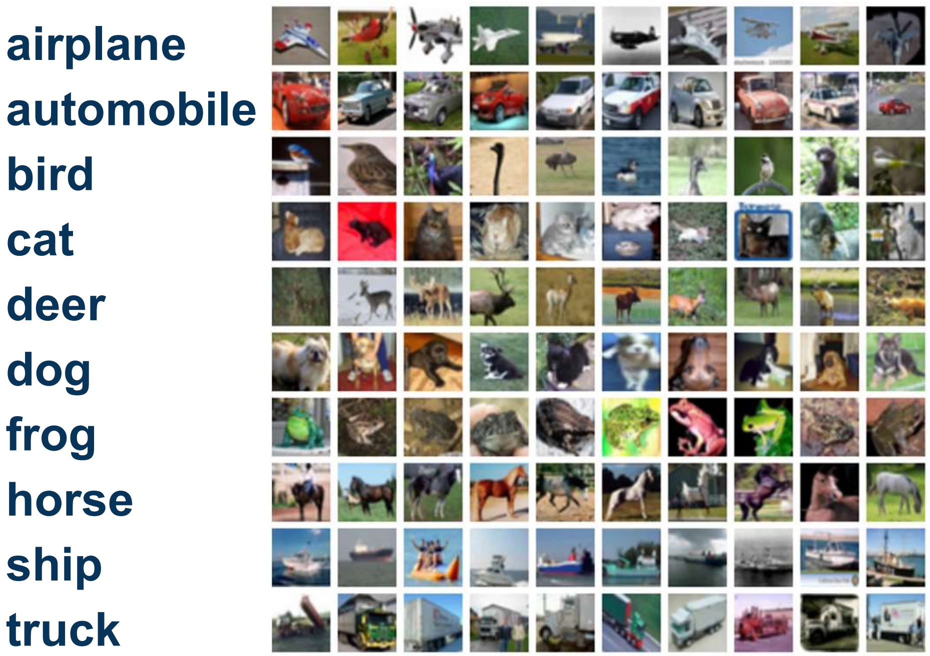
Interpreting a Linear Classifier



Visual Viewpoint

We can convert the weight vector back into the shape of the image and visualize

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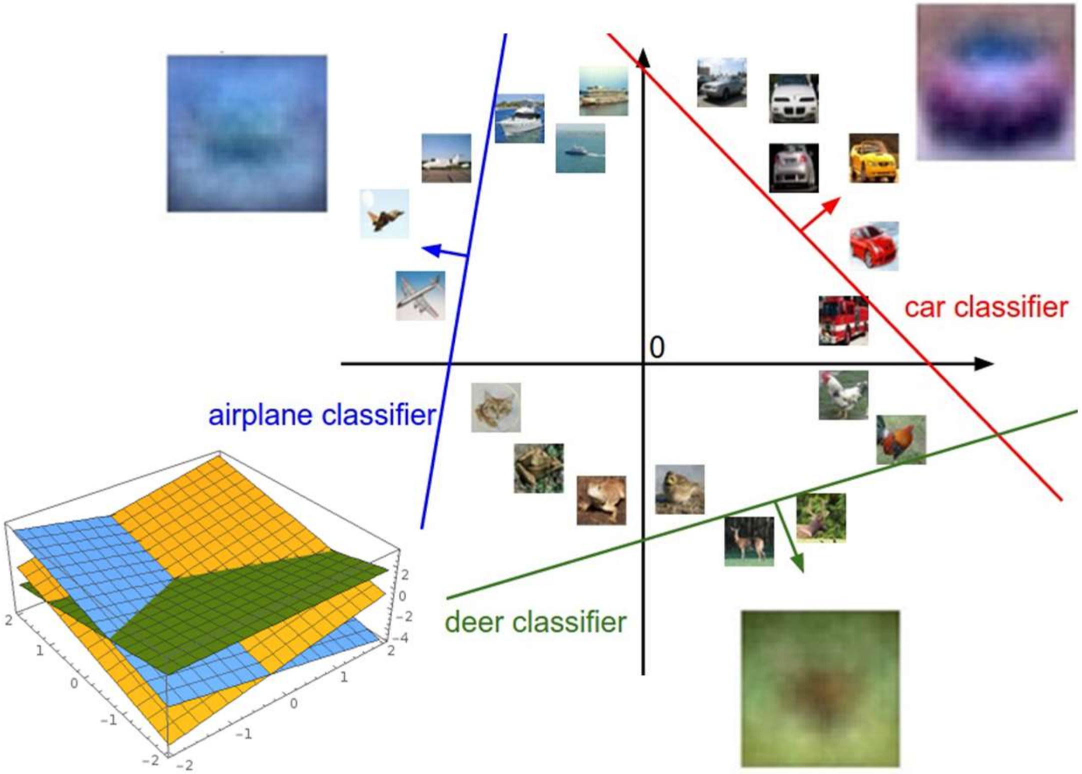
Geometric Viewpoint



Plot created using Wolfram Cloud

$$f(x, W) = Wx + b$$

Array of **32x32x3** numbers
(3072 numbers total)

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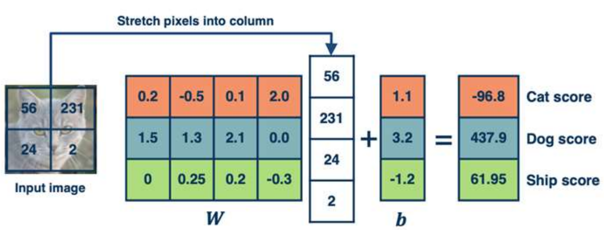
Interpreting a Linear Classifier

Algebraic Viewpoint

$$f(x, W) = Wx$$

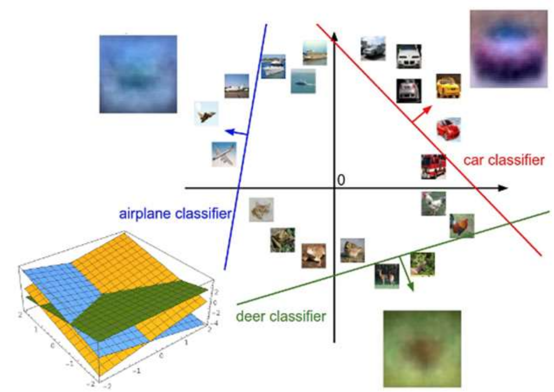


Visual Viewpoint

One template per class

Geometric Viewpoint

Hyperplanes cutting up space



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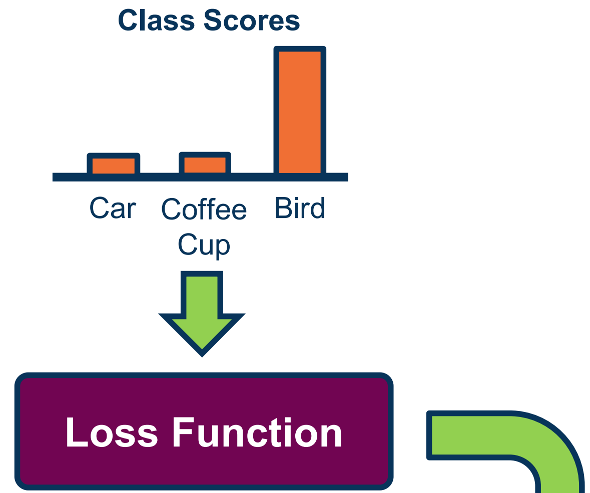
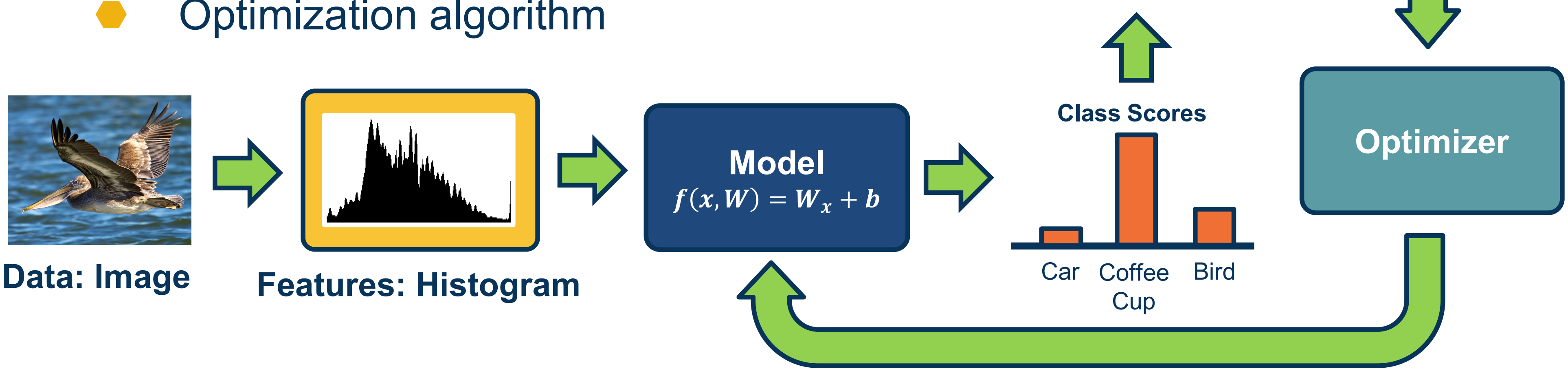
Linear Classifier: Three Viewpoints

Performance Measure for a Classifier

- Input (and representation)
- Functional form of the model
  - Including parameters
- **Performance measure to improve**
  - **Loss or objective function**
- Algorithm for finding best parameters
  - Optimization algorithm



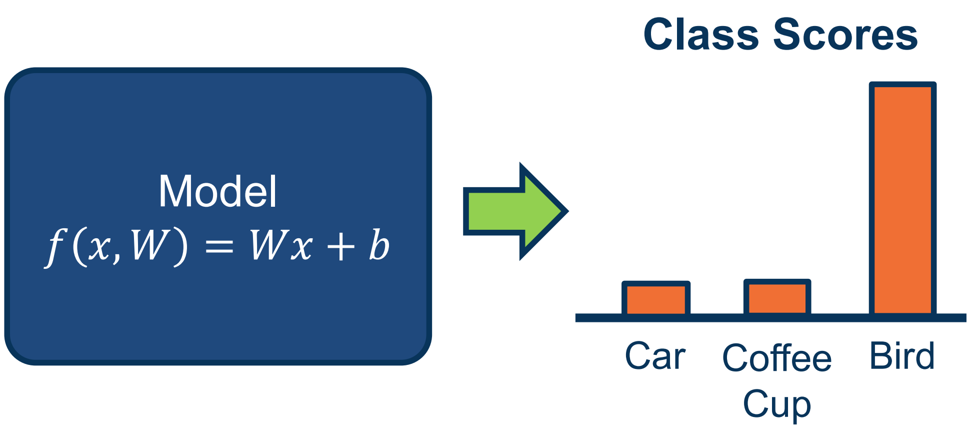


Components of a Parametric Model

- The output of a classifier can be considered a **score**
- For binary classifier, use rule:

$$y = \begin{cases} 1 & \text{if } f(x, w) >= 0 \\ 0 & \text{otherwise} \end{cases}$$

  - Can be used for many classes by considering one class versus all the rest (one versus all)
- For multi-class classifier can take the maximum



Classification using Scores

Several issues with scores:

- Not very interpretable (no bounded value)

We often want **probabilities**

- More interpretable
- Can relate to probabilistic view of machine learning

We use the **softmax** function to convert scores to probabilities

$$s = f(x, W)$$ **Scores**

$$P(Y = k | X = x) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$ **Softmax Function**

Converting Scores to Probabilities

Performance Measure

We need a performance measure to **optimize**

- Penalizes model for being wrong
- Allows us to modify the model to reduce this penalty
- Known as an **objective** or **loss** function

In machine learning we use **empirical risk minimization**

- Reduce the loss over the **training** dataset
- We **average** the loss over the training data

Given a dataset of examples:

$$\{(x_i, y_i)\}_{i=1}^{N}$$

Where x_i is image and

y_i is (integer) label

Loss over the dataset is a sum of loss over examples:

$$L = \frac{1}{N} \sum L(f(x_i, W), y_i)$$

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Multiclass SVM loss:

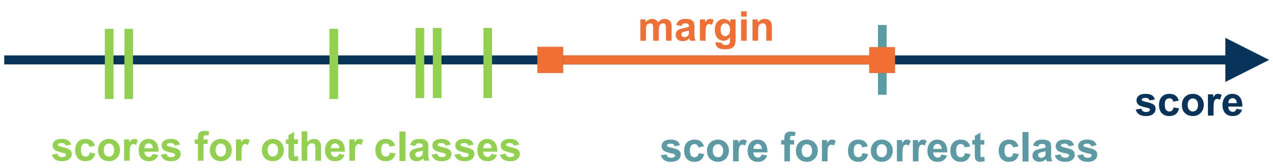
Given an example (x_i, y_i) where x_i is the image and where y_i is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

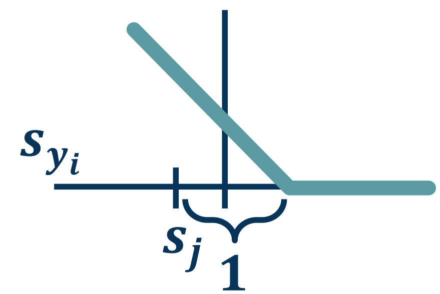
the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \begin{cases} 0 & \text{if } s_{y_i} \geq s_j + 1 \\ s_j - s_{y_i} + 1 & \text{otherwise} \end{cases}$$

$$= \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$



Example: "Hinge Loss"



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Multiclass SVM loss:

Given an example $(\boldsymbol{x}_{i,}\boldsymbol{y}_i)$ where $\boldsymbol{x}_i$ is the image and where $\boldsymbol{y}_i$ is the (integer) label,

and using the shorthand for the scores vector: $\boldsymbol{s} = \boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{W})$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

= max(0, 5.1 - 3.2 + 1)
+max(0, -1.7 - 3.2 + 1)
= max(0, 2.9) + max(0, -3.9)
= 2.9 + 0
= 2.9

Suppose: 3 training examples, 3 classes.
With some $\boldsymbol{W}$ the scores $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{W})=\boldsymbol{W}\boldsymbol{x}$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	**2.9**		

Adapted from slides by Fei-Fei Li, Justin Johnson, Serena Yeung, from CS 231n

Multiclass SVM loss:

Given an example (x_i, y_i) where x_i is the image and where y_i is the (integer) label,

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the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

= max(0, 1.3 - 4.9 + 1)
+max(0, 2.0 - 4.9 + 1)
= max(0, -2.6) + max(0, -1.9)
= 0 + 0
= 0

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:		**0.0**	

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SVM Loss Example

Multiclass SVM loss:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

Q: What happens to loss if car image scores change a bit?

No change for small values

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

  

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
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Multiclass SVM loss:

Suppose: 3 training examples, 3 classes.
With some $\boldsymbol{W}$ the scores $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{W})=\boldsymbol{W}\boldsymbol{x}$ are:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

Q: What is min/max of loss value?

[0,inf]

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**

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SVM Loss Example

Multiclass SVM loss:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

Q: At initialization W is small so all s ≈ 0.
What is the loss?

C-1

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**

Adapted from slides by Fei-Fei Li, Justin Johnson, Serena Yeung, from CS 231n

Multiclass SVM loss:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

Q: What if the sum was over all classes? (including j = y_i)

No difference (add constant 1)

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**

Adapted from slides by Fei-Fei Li, Justin Johnson, Serena Yeung, from CS 231n

SVM Loss Example

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Multiclass SVM loss:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

Q: What if we used mean instead of sum?

No difference
Scaling by constant

Suppose: 3 training examples, 3 classes.
With some $\boldsymbol{W}$ the scores $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{W})=\boldsymbol{W}\boldsymbol{x}$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**

Adapted from slides by Fei-Fei Li, Justin Johnson, Serena Yeung, from CS 231n

SVM Loss Example

Multiclass SVM loss:

Given an example (x_i, y_i) where x_i is the image and where y_i is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

L = (2.9 + 0 + 12.9)/3
= **5.27**

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	12.9

Adapted from slides by Fei-Fei Li, Justin Johnson, Serena Yeung, from CS 231n

SVM Loss Example

- If we use the softmax function to convert scores to probabilities, the right loss function to use is **cross-entropy**
- Can be derived by looking at the distance between two probability distributions (output of model and ground truth)
- Can also be derived from a maximum likelihood estimation perspective

$$s = f(x, W)$$ **Scores**

$$P(Y = k | X = x) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$ **Softmax Function**

$$L_i = -\log P(Y = y_i | X = x_i)$$

Maximize log-prob of correct class = Maximize the log likelihood = Minimize the negative log likelihood

Performance Measure for Probabilities

- If we use the softmax function to convert scores to probabilities, the right loss function to use is **cross-entropy**
- Goal: Minimize KL-divergence (distance measure b/w probability distributions)

$$\min_w KL(p^*||\hat{p}) = \sum_y p^*(y)\ log \frac{p^*(y)}{\hat{p}(y)}$$

$$= \sum_y p^*(y) \log(p^*(y)) - \sum_y p^*(y) \log(\hat{p}(y))$$

$-H(p^*)$
(negative entropy, term goes away because not a function of model, *W*, parameters we are minimizing over)

$H(p^*, \hat{p})$
(Cross-Entropy)

$$p^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{p} = \begin{bmatrix} P(Y=1|x,w) \\ P(Y=2|x,w) \\ P(Y=3|x,w) \\ P(Y=4|x,w) \\ P(Y=5|x,w) \\ P(Y=6|x,w) \\ P(Y=7|x,w) \\ P(Y=8|x,w) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.15 \\ 0.3 \end{bmatrix}$$

Ground Truth **Prediction**

Since p^* is one-hot (0 for non-ground truth classes), all we need to minimize is (where i is ground truth class): $\min_w (-log\ \hat{p}(y_i))$

Performance Measure for Probabilities

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax Function

$$L_i = -\log P(Y = y_i | X = x_i)$$

Probabilities must be >= 0

Probabilities must sum to 1

	Unnormalized log-probabilities / logits		Unnormalized probabilities		Probabilities
cat	**3.2**	exp →	**24.5**	normalize →	**0.13**
car	5.1		164.0		0.87
frog	-1.7		0.18		0.00

→ $L_i = -\log(0.13)$

Adapted from slides by Fei-Fei Li, Justin Johnson, Serena Yeung, from CS 231n

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

Probabilities must be >= 0

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax Function

Probabilities must sum to 1

$$L_i = -\log P(Y = y_i | X = x_i)$$

$$L_i = -\log(0.13)$$

Q: What is the min/max of possible loss L_i?

Infimum is 0, max is unbounded (inf)

Adapted from slides by Fei-Fei Li, Justin Johnson, Serena Yeung, from CS 231n

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax Function

Probabilities must be >= 0

Probabilities must sum to 1

$$L_i = -\log P(Y = y_i | X = x_i)$$

$$L_i = -\log(0.13)$$

Q: At initialization all s will be approximately equal; what is the loss?

Log(C), e.g. log(10) ≈ 2

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Often, we add a **regularization term** to the loss function

L1 Regularization

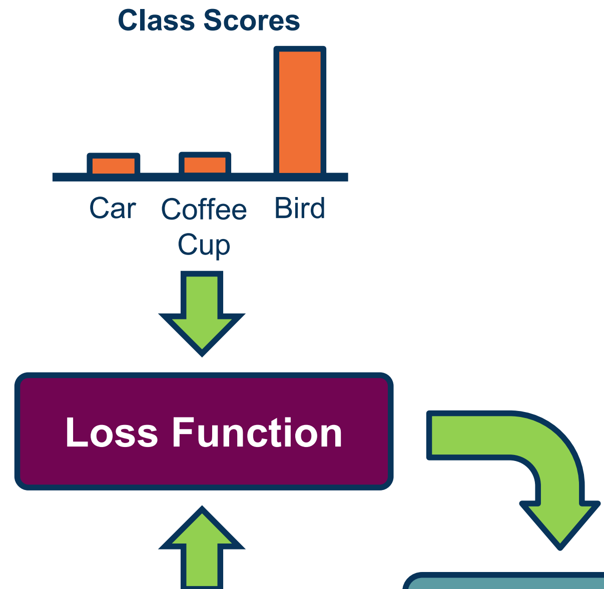
$$L_i = |y - Wx_i|^2 + |W|$$

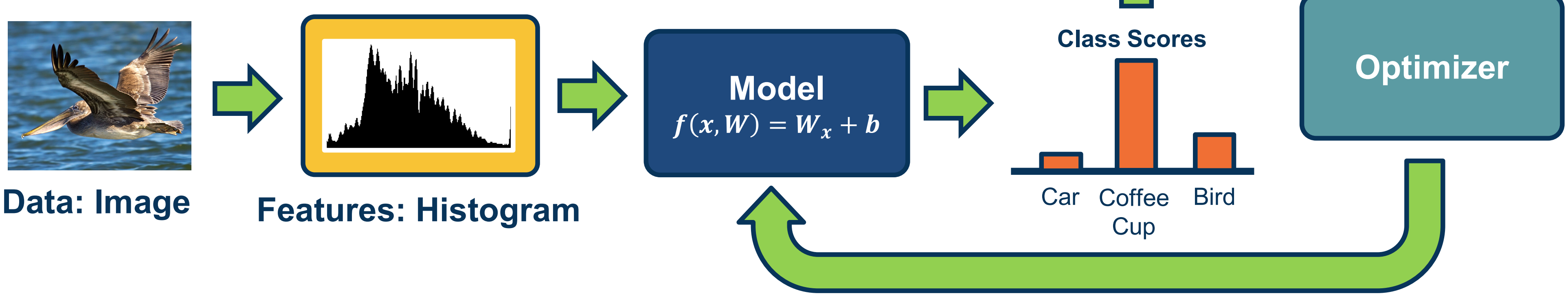
Example regularizations:

- L1/L2 on weights (encourage small values)

Gradient Descent

- Input (and representation)
- Functional form of the model
  - Including parameters
- Performance measure to improve
  - Loss or objective function
- **Algorithm for finding best parameters**
  - **Optimization algorithm**





Components of a Parametric Model

Optimization

Given a model and loss function, finding the best set of weights is a **search problem**

- Find the best combination of weights that minimizes our loss function

Several classes of methods:

- Random search
- Genetic algorithms (population-based search)
- Gradient-based optimization

In deep learning, **gradient-based methods are dominant** although not the only approach possible

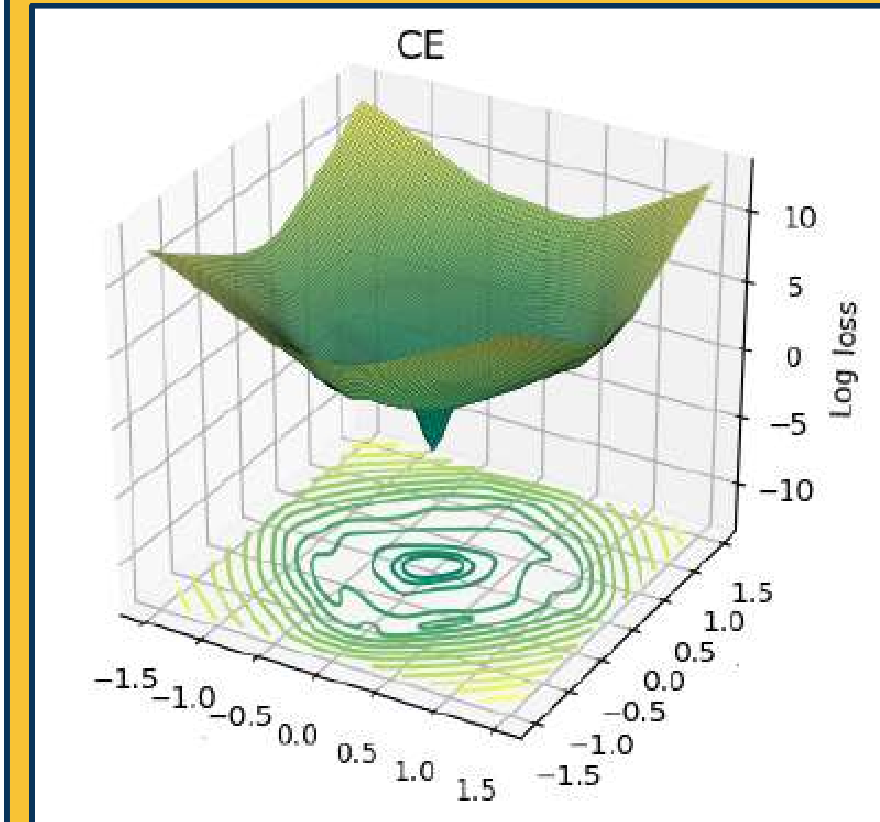
$$\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1m} & b1 \\ w_{21} & w_{22} & \cdots & w_{2m} & b2 \\ w_{21} & w_{22} & \cdots & w_{3m} & b3 \end{bmatrix}$$

Loss

As weights change, the loss changes as well

- This is often somewhat-smooth locally, so small changes in weights produce small changes in the loss

We can therefore think about **iterative algorithms that take current values of weights and modify them** a bit



Strategy: Follow the Slope!

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- We can find the steepest descent direction by computing the **derivative (gradient):**

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Steepest descent direction is the **negative gradient**
- **Intuitively:** Measures how the function changes as the argument a changes by a small step size
  - As step size goes to zero
- **In Machine Learning:** Want to know how the **loss function** changes **as weights** are varied
  - Can consider each parameter separately by taking **partial derivative** of loss function with respect to that parameter

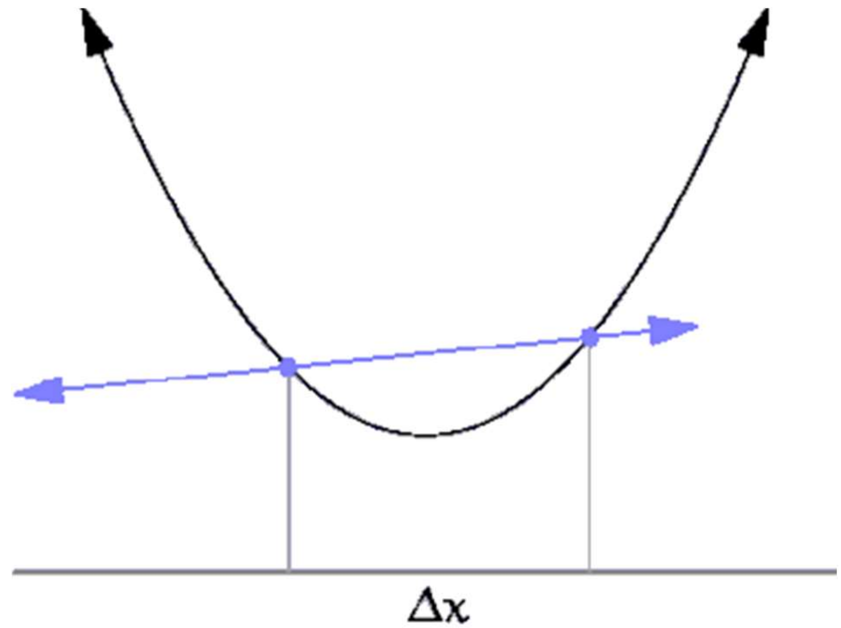


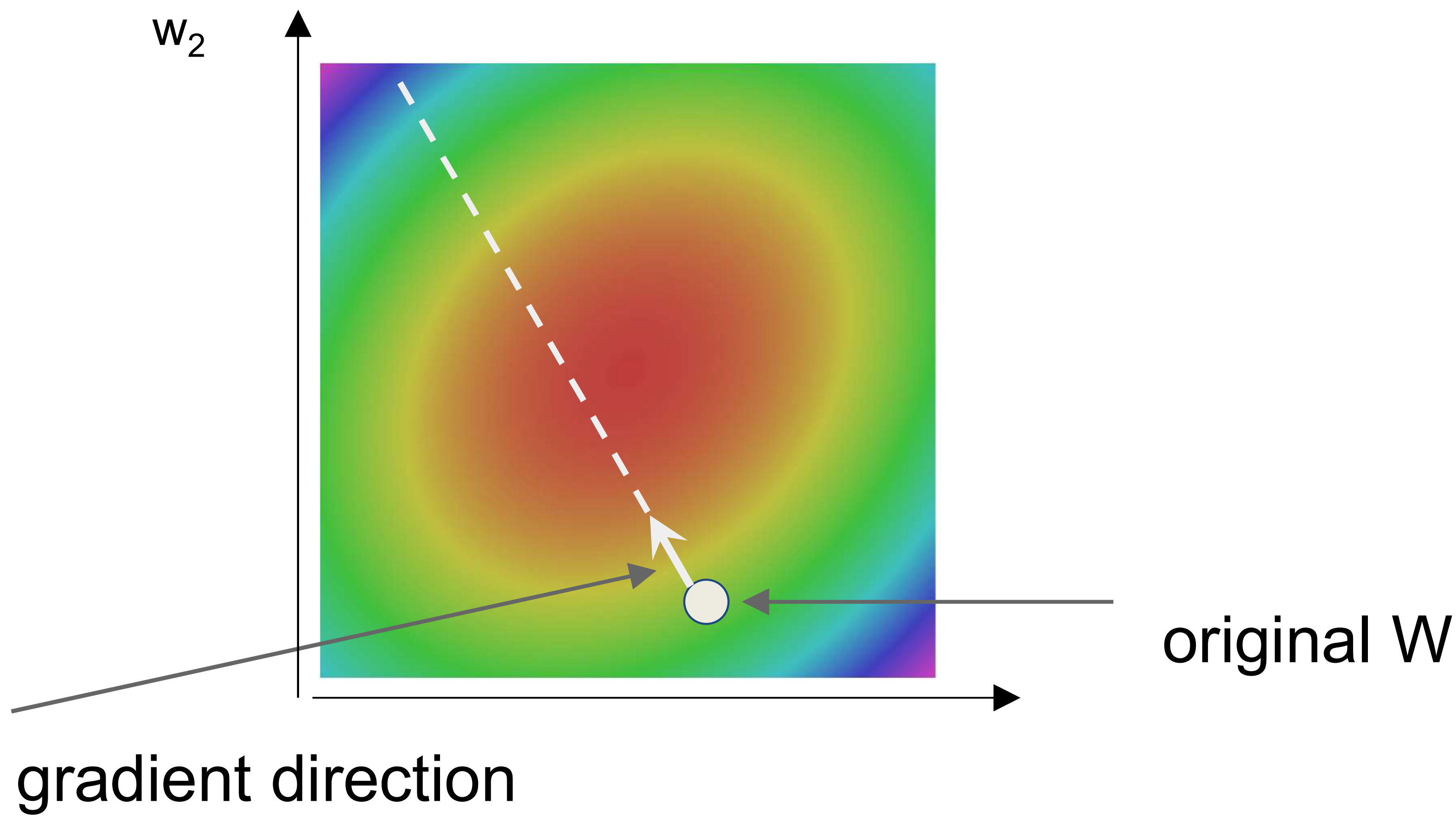
Image and equation from: https://en.wikipedia.org/wiki/Derivative#/media/File:Tangent_animation.gif

Derivatives

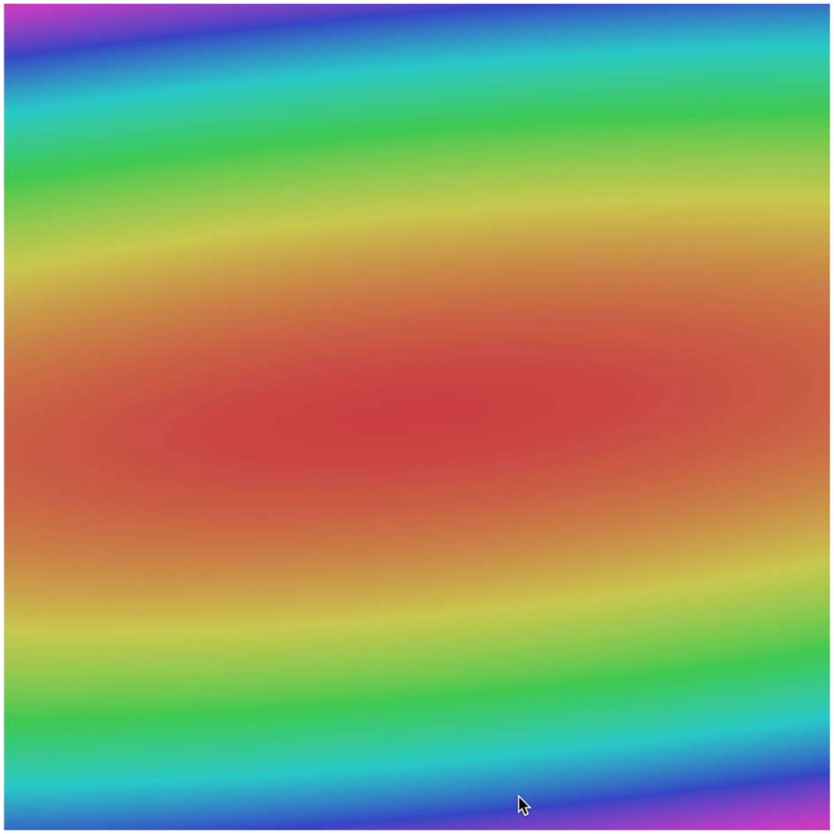
This idea can be turned into an **algorithm (gradient descent)**

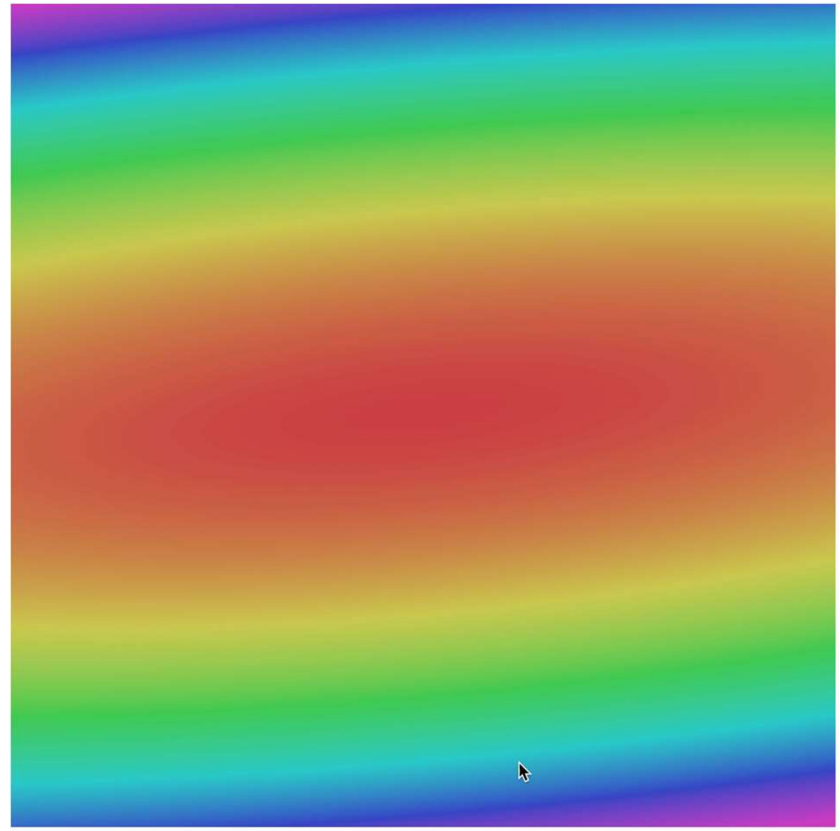
- Choose a model: $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{W}) = \mathrm{Wx}$
- Choose loss function: $\boldsymbol{L_i} = (\boldsymbol{y} - \boldsymbol{W}\boldsymbol{x_i})^2$
- Calculate partial derivative for each parameter: $\frac{\partial L}{\partial w_i}$
- Update the parameters: $w_i = w_i - \frac{\partial L}{\partial w_i}$
- Add learning rate to prevent too big of a step: $w_i = w_i - \alpha \frac{\partial L}{\partial w_i}$
- Repeat (from Step 3)

http://demonstrations.wolfram.com/VisualizingTheGradientVector/



Gradient Descent





w_1

Gradient Descent

Often, we only compute the gradients across a small subset of data

- Full Batch Gradient Descent $$\boldsymbol{L} = \frac{1}{N}\sum \boldsymbol{L}\left(\boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{W}), \boldsymbol{y}_i\right)$$

- Mini-Batch Gradient Descent $$\boldsymbol{L} = \frac{1}{M}\sum \boldsymbol{L}\left(\boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{W}), \boldsymbol{y}_i\right)$$
  - Where M is a *subset* of data
- We iterate over mini-batches:
  - Get mini-batch, compute loss, compute derivatives, and take a set

Mini-Batch Gradient Descent

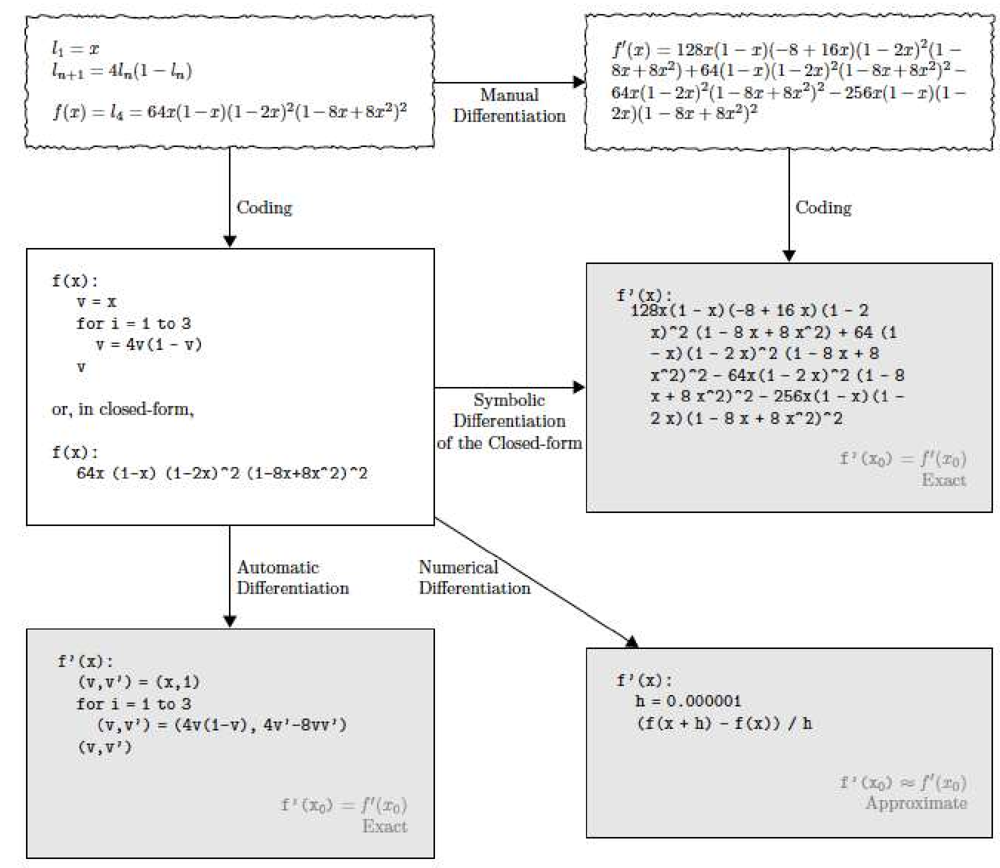
Gradient descent is guaranteed to converge under some conditions

- For example, learning rate has to be appropriately reduced throughout training
- It will converge to a *local* minima
  - Small changes in weights would not decrease the loss
- It turns out that some of the local minima that it finds in practice (if trained well) are still pretty good!

We know how to compute the **model output and loss function**

Several ways to compute $\frac{\partial L}{\partial w_i}$

- Manual differentiation
- Symbolic differentiation
- Numerical differentiation
- Automatic differentiation



current W:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
loss 1.25347

gradient dW:

[?,
?,
?,
?,
?,
?,
?,
?,
?,…]

current W:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
loss 1.25347

W + h (first dim)**:**

[0.34 + **0.0001**,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
loss 1.25322

gradient dW:

[?,
?,
?,
?,
?,
?,
?,
?,
?,…]

current W:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
loss 1.25347

W + h (first dim)**:**

[0.34 + **0.0001**,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
loss 1.25322

gradient dW:

[**-2.5**,
?,
?,

(1.25322 - 1.25347)/0.0001
= -2.5

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

?,
?,…]

current W:	**W + h** (second dim)**:**	**gradient dW:**
[0.34,	[0.34,	[-2.5,
-1.11,	-1.11 + **0.0001**,	?,
0.78,	0.78,	?,
0.12,	0.12,	?,
0.55,	0.55,	?,
2.81,	2.81,	?,
-3.1,	-3.1,	?,
-1.5,	-1.5,	?,
0.33,…]	0.33,…]	?,…]
loss 1.25347	**loss 1.25353**	

current W:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
loss 1.25347

W + h (second dim)**:**

[0.34,
-1.11 + **0.0001**,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
loss 1.25353

gradient dW:

[-2.5,
0.6,
?,
?,

(1.25353 - 1.25347)/0.0001
= 0.6

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

?,…]

current W:	**W + h** (third dim)**:**	**gradient dW:**
[0.34,	[0.34,	[-2.5,
-1.11,	-1.11,	0.6,
0.78,	0.78 + **0.0001**,	?,
0.12,	0.12,	?,
0.55,	0.55,	?,
2.81,	2.81,	?,
-3.1,	-3.1,	?,
-1.5,	-1.5,	?,
0.33,…]	0.33,…]	?,…]
loss 1.25347	**loss 1.25347**	

current W:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
loss 1.25347

W + h (third dim)**:**

[0.34,
-1.11,
0.78 + **0.0001**,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
loss 1.25347

gradient dW:

[-2.5,
0.6,
0,
?,
?,…]

(1.25347 - 1.25347)/0.0001
= 0

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Numerical vs Analytic Gradients

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Numerical gradient: slow :(, approximate :(, easy to write :)
Analytic gradient: fast :), exact :), error-prone :(

In practice: Derive analytic gradient, check your implementation with numerical gradient.
This is called a **gradient check.**

Summary

- Components of parametric classifiers:
  - Input/Output: Image/Label
  - Model (function): Linear Classifier + Softmax
  - Loss function: Cross-Entropy
  - Optimizer: Gradient Descent

- Ways to compute gradients
  - Numerical
  - Next: Analytical, automatic differentiation

For some functions, we can analytically derive the partial derivative

Example:

Function

$$f(\boldsymbol{w}, \boldsymbol{x}_i) = \boldsymbol{w}^T \boldsymbol{x}_i$$

Loss

$$(y_i - \boldsymbol{w}^T \boldsymbol{x}_i)^2$$

(Assume $\boldsymbol{w}$ and $\mathbf{x_i}$ are column vectors, so same as $\boldsymbol{w} \cdot \boldsymbol{x}_i$)

Update Rule

$$w_j \leftarrow w_j + 2\eta \sum_{k=1}^{N} \delta_k x_{kj}$$

Derivation of Update Rule

For some functions, we can analytically derive the partial derivative

Example:

Function

$$f(w, x_i) = w^T x_i$$

Loss

$$(y_i - w^T x_i)^2$$

(Assume w and $\mathbf{x_i}$ are column vectors, so same as $w \cdot x_i$)

Update Rule

$$w_j \leftarrow w_j + 2\eta \sum_{k=1}^{N} \delta_k x_{kj}$$

Derivation of Update Rule

L$= \sum_{k=1}^{N} (y_k - w^T x_k)^2$

Gradient descent tells us we should update w as follows to minimize L:

$$w_j \leftarrow w_j - \eta \frac{\partial L}{\partial w_j}$$

So what's $\frac{\partial L}{\partial w_j}$?

$$\frac{\partial L}{\partial w_j} = \sum_{k=1}^{N} \frac{\partial}{\partial w_j} (y_k - w^T x_k)^2$$

$$= \sum_{k=1}^{N} 2(y_k - w^T x_k) \frac{\partial}{\partial w_j} (y_k - w^T x_k)$$

$$= -2 \sum_{k=1}^{N} \delta_k \frac{\partial}{\partial w_j} w^T x_k$$

…where…
$\delta_k = y_k - w^T x_k$

$$= -2 \sum_{k=1}^{N} \delta_k \frac{\partial}{\partial w_j} \sum_{i=1}^{m} w_i x_{ki}$$

$$= -2 \sum_{k=1}^{N} \delta_k x_{kj}$$

If we add a **non-linearity (sigmoid),** derivation is more complex

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

First, one can derive that: $\sigma'(x) = \sigma(x)(1-\sigma(x))$

$$\mathrm{f}(\mathrm{x}) = \sigma\left(\sum_k w_k x_k\right)$$

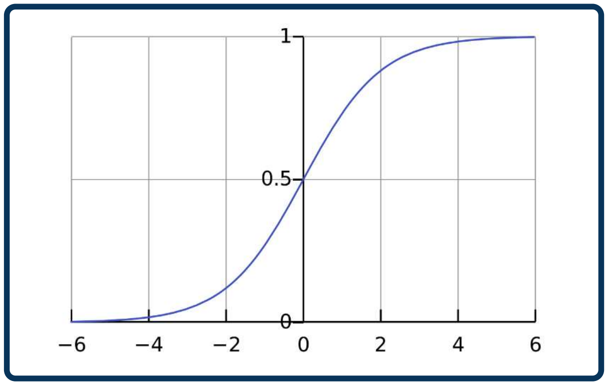
$$\mathrm{L} = \sum_i \left(y_i - \sigma\left(\sum_k w_k x_{ik}\right)\right)^2$$

$$\frac{\partial L}{\partial w_j} = \sum_i 2\left(y_i - \sigma\left(\sum_k w_k x_{ik}\right)\right)\left(-\frac{\partial}{\partial w_j}\sigma\left(\sum_k w_k x_{ik}\right)\right)$$

$$= \sum_i -2\left(y_i - \sigma\left(\sum_k w_k x_{ik}\right)\right)\sigma'\left(\sum_k w_k x_{ik}\right)\frac{\partial}{\partial w_j}\sum_k w_k x_{ik}$$

$$= \sum_i -2\delta_i \sigma(\mathrm{d}_i)(1-\sigma(\mathrm{d}_i))x_{ij}$$

where $\delta_i = y_i - \mathrm{f}(x_i)$ $\quad d_i = \sum w_k x_{ik}$



The sigmoid perception update rule:

$$w_j \leftarrow w_j + 2\eta \sum_{k=1}^{N} \delta_i \sigma_i (1-\sigma_i) x_{ij}$$

where $\sigma_i = \sigma\left(\sum_{j=1}^{m} w_j x_{ij}\right)$

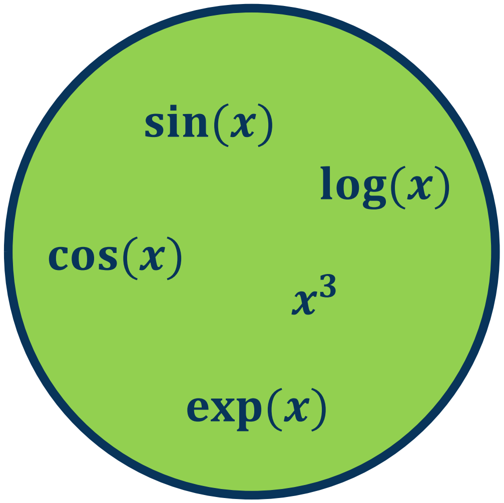
$$\delta_i = y_i - \sigma_i$$

Adding a Non-Linear Function

Georgia Tech

Decomposing a Function

Given a library of simple functions



Compose into a complicate function

$$-\log\left(\frac{1}{1+e^{-w\cdot x}}\right)$$

Adapted from slides by: Marc'Aurelio Ranzato, Yann LeCun